

# Analysis 1

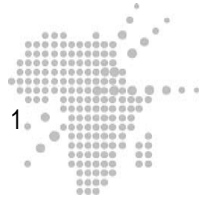
## Analysis 1



Prepared by Jairus M. KHALAGAI



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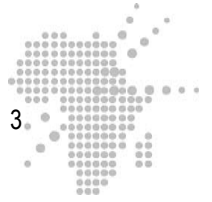
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## **I. Analysis 1**

by Prof. Jairus M. Khalagai

## **II. Prerequisite Courses Or Knowledge**

### **Unit 1 : Analysis of the real line**

Basic mathematics and calculus

### **Unit 2 : Vector Analysis**

Calculus

### **Unit 3 : Complex Analysis**

Calculus and Analysis unit 1

## **III. Time**

The total time for this module is 120 study hours.

## **IV. Material**

Students should have access to the core readings specified later. Also, they will need a computer to gain full access to the core readings. Additionally, students should be able to install the computer software wxMaxima and use it to practice algebraic concepts.

## **V. Module Rationale**

The rationale of teaching analysis is to set the minimum content of Pure Mathematics required at undergraduate level for student of mathematics. It is important to note that skill in proving mathematical statements is one aspect that learners of Mathematics should acquire. The ability to give a complete and clear proof of a theorem is essential for the learner so that he or she can finally get to full details and rigor of analyzing mathematical concepts. Indeed it is in Analysis that the learner is given the exposition of subject matter as well as the techniques of proof equally. We also note here that if a course like calculus with its wide applications in Mathematical sciences is an end in itself then Analysis is the means by which we get to that end.



## VI. Content

### 6.1 Overview

This module consists of three units which are as follows:

#### Unit 1 - Analysis on the real line

In this unit we start by decomposing the set  $\mathbb{R}$  of real numbers into its subsets. We then define the so called standard metric on  $\mathbb{R}$  to be able to study its structure which consists of concepts like open and closed intervals, neighbourhoods, interior and limit points leading to examples of open and closed subsets of  $\mathbb{R}$ . Countability of such subsets and numerical sequences are also essential in the structure of  $\mathbb{R}$  as a metric space. We also study functions defined on sets of real numbers with respect to concepts of continuity differentiability and integrability.

#### Unit 2 – Vector Analysis

This unit deals with mainly vector calculus and its applications. Thus we specifically consider concepts like gradient, divergence and curl. This leads to well known theorems like Green's stokes divergence theorems and other related results. The curvilinear coordinates system is also given a treat in this unit.

#### Unit 3 – Complex Analysis

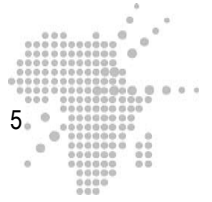
The main objective in the study of this unit is to define a function of a complex variable and then look at its degree of smoothness such as existence of limit, continuity and differentiability along the lines of calculus of functions of a real variable. Also a look at integration and power series involving a function of a complex variable will complete this unit.

### 6.2 Outline: Syllabus

#### Unit 1a – Introduction to Analysis (25 hours)

Level priority Basic Mathematics and Calculus are Pre-requisites

- Real number system and completeness
- Open, closed intervals and neighbourhoods
- Interior points, limit points
- Uncountability of the real line
- Sequences
- Functions, limits and continuity
- Differentiability and integrability

**Unit 1a – The Riemann Integral (15 hours)**

- Define the Riemann integral
- Give examples of Riemann integrals defined on the set  $\mathbb{R}$
- State some properties of a Riemann integral
- Verify or prove some results on Riemann integrals.

**Unit 2 – Vector Analysis (40 hours)**

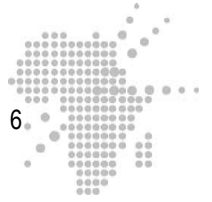
Level priority Basic Mathematics and Calculus are Pre-requisites

- Gradient, divergence and Curl
- Green's, Stokes and divergence theorems
- Other related theorems
- Curvilinear coordinates

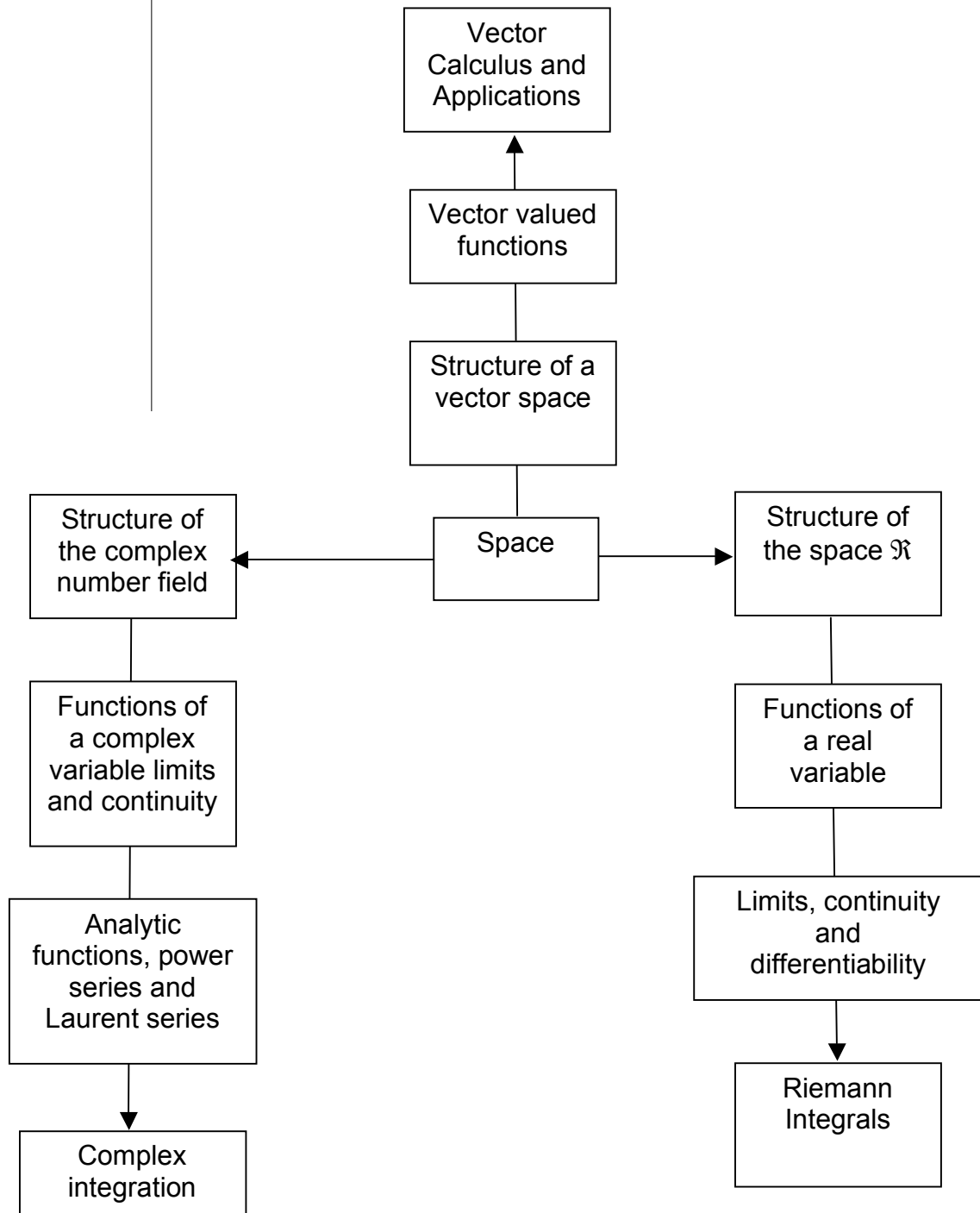
**Unit 3 – Complex Analysis (40 hours)**

Level priority Unit 1 is the Pre-requisite

- Definition and examples of a complex variable
- Limits, continuity and differentiability
- Analytic functions
- Complex integration
- Power and Laurent series



### 6.3 Graphic Organiser





## **VII. Specific Learning Objectives (Instructional Objectives)**

You should be able to:

1. Demonstrate understanding of basic concepts and principles of mathematical analysis.
2. Develop a logical framework for stating and proving theorems.





## VIII. Teaching And Learning Activities

### Pre-assessment

Q1. Given the function

$$F(x, y) = x^3 y^2 \text{ find } \frac{2F}{2x}$$

- a)  $2x^3 y$
- b)  $3x^2 y^2$
- c)  $6x^2 y$
- d)  $3x^2 + 2y$

Q2. Given two vectors

$$\underline{u} = 3\underline{i} + 4\underline{j} - 2\underline{k}$$

$$\underline{v} = -5\underline{i} + 2\underline{j} + 3\underline{k}$$

Find  $\underline{u} \cdot \underline{v}$

- a)  $-3$
- b)  $-13$
- c)  $17$
- d)  $-17$

Q3. For the vectors  $\underline{a} = 2\underline{c} - 3\underline{j} + \underline{k}$  and

$$\underline{b} = -3\underline{i} + 4\underline{j} + 2\underline{k} \text{ find } \underline{a} \times \underline{b}$$

- a)  $-10\underline{i} + 7\underline{j} + \underline{k}$
- b)  $-10\underline{i} - 7\underline{j} + \underline{k}$
- c)  $-10\underline{i} - 7\underline{j} - \underline{k}$
- d)  $-10\underline{i} - 7\underline{j} + \underline{k}$



Q4. Given the complex numbers

$$Z_1 = 3 + 4i \text{ and } Z_2 = -2 + 3i$$

find  $Z_1 + Z_2$

a)  $\sqrt{150}$       b)  $\sqrt{24}$       c)  $\sqrt{5}$       d)  $\sqrt{38}$

Q5. Given two complex numbers

$$Z_1 = 5 + 2i \text{ and } Z_2 = 4 + 3i$$

find  $\text{Arg } Z_1 Z_2$

a)  $\tan^{-1} \frac{2}{5}$       b)  $\tan^{-1} \frac{23}{14}$       c)  $\tan^{-1} \frac{3}{4}$       d)  $\tan^{-1} \frac{15}{9}$

Q6. Evaluate

$$\frac{1 + 2i}{3 - 4i} + \frac{2 - i}{5i}$$

a)  $\frac{-2}{5}$       b)  $\frac{5}{2}$       c) 1      d)  $\frac{3 + i}{3 - i}$

Q7. Express the complex number

$$Z = 1 + i \text{ in modulus argument form}$$

a)  $Z = \sqrt{3} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$

b)  $Z = \sqrt{2} \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$

c)  $Z = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$

d)  $Z = \sqrt{3} \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$



Q.8 Which of the following functions is not continuous at  $x = 0$ ?

a)  $f(x) = x^2$

b)  $g(x) = 1 + x^2$

c)  $h(x) = |x|$

d) None

Q9. Which of these functions is not different at  $x = 0$ ?

a)  $f(x) = x^2$

b)  $g(x) = 1 + x^2$

c)  $h(x) = |x|$

d) None

Q10. Which one of the following statements is true about the number  $\sqrt{2}$ ?

a)  $\sqrt{2}$  is an integer

b) It is a fraction

c) It is a rational

d) It is irrational

### Pre-assessment: Solutions

Q 1. (b)

Q 2. (b)

Q 3. (c)

Q 4. (a)

Q 5. (b)

Q 6. (a)

Q 7. (c)

Q 8. (d)

Q 7. (c)



## IX. Learning Activities

### Analysis 1: Unit 1a

**Title:** Introduction to Analysis

#### Specific Objectives

At the end of this activity the learner will be able to:

- Decompose the set  $\mathfrak{R}$  of real numbers into various subsets.
- State properties of various subsets of  $\mathfrak{R}$ .
- Define the concept of boundedness of sets of real numbers.
- State and use the completeness axiom to solve problems on real number system.

#### Summary

We first note that a satisfactory discussion of the main concepts of Analysis (e.g. convergence, continuity, differentiability and integration), must be based on an accurately defined number concept. To this end the real number system is indeed a well defined concept. It is clear here that the axioms governing the arithmetic of the integers are well known and trivial at this level. However, we need to familiarize

ourselves with the arithmetic of rational numbers (i.e. numbers of the form  $\frac{n}{m}$  where  $n$  and  $m$  are integers  $m \neq 0$ ) and we shall list the main feature of the arithmetic of such numbers as that of being closed under both addition and multiplication.

We also note that a relation  $<$  is defined and brings order on the set of rational numbers. Thus for any two rational numbers  $p$  and  $q$  we have that either  $p = q$  or  $q < p$  and it is transitive. It is also known that the rational number system is inadequate. For instance there is no rational number  $p$  such that  $p^2 = 2$ .

This leads to the introduction of the set of irrational numbers. Thus the union of the set of rational numbers and the set of irrational numbers constitutes the set  $\mathfrak{R}$  of real numbers. We will also consider the concept of a set of real numbers being bounded below or above. This leads to the completeness axiom that characterizes the set  $\mathfrak{R}$  of real numbers completely.



## Key Concepts

- Rational number:** This is a number that can be written in the form  $\frac{n}{m}$  where  $n$  and  $m$  are integers with  $m \neq 0$  and 1 as the only common factor for  $n$  and  $m$ .
- Irrational number:** This is a number that cannot be written in the form  $\frac{n}{m}$ , where  $n$  and  $m$  are integers with  $m \neq 0$ . Thus, we have that the set of rational numbers is  $\mathbb{Q}$  and the set of irrational numbers  $\mathbb{Q}^c$  leading to the relation  $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$  i.e. Every real number is either rational or irrational
- Bounded below:** Any set  $E$  of real numbers is said to be bounded below if there is a real number  $q$  such that  $q \leq x \quad \forall x \in E$ . In this case  $q$  is called a lower bound for  $E$ .
- Bounded above:** Any set  $E$  of real numbers is said to be bounded above if there is a real number  $p$  such that  $p \geq x \quad \forall x \in E$ . In this case  $p$  is called an upper bound for  $E$ .
- Infimum:** For any set  $E$  of real numbers, the largest of all real numbers  $q$  such that  $q \leq x \quad \forall x \in E$ . is called infimum of  $E$  written  $\text{Inf } E$  or is called greatest lower bound written  $\text{glb } E$ .
- Supremum:** For any set  $E$  of real numbers the smallest of all real numbers  $p$  such that  $p \geq x \quad \forall x \in E$  is called the supremum of  $E$  written  $\text{sup } E$  or is called the least upper bound of  $E$  written  $\text{lub } E$ .
- Completeness axiom:** This states as follows: Every non empty set  $E$  of real numbers bounded above has supremum and if bounded below has infimum.



## 1a. Learning Activity: The Real Number System.

### Story of Siamese twins

The case of siamese twins in Kenya was first reported in 1991. These were twins both boys born at Kenyatta National Hospital with their bodies attached to each other at the chest. This meant that they were sharing the same heart. An operation was carried out later to separate them was unsuccessful. Indeed to be able to separate Siamese twins through operation depends on the organ they are sharing.

### Question

Have you heard of such case of siamese twins in other animals, say cows?

### 1a.0 Introduction

We now recall that the real line  $\mathbb{R}$  can be decomposed into the following subsets.

$\mathbb{N} = \{1, 2, \dots\}$  as the set of natural numbers.

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$  as the set of integers.

$\mathbb{Q} = \left\{ r : r = \frac{n}{m}, n, m \in \mathbb{Z}, m \neq 0, (n, m) = 1 \right\}$  as the set of rational numbers.

$\mathbb{I} = \left\{ r : r \neq \frac{n}{m}, n, m \in \mathbb{Z}, m \neq 0, (n, m) = 1 \right\}$  as the set of irrational numbers.

### Remarks

(i) Note that here  $(n, m) = 1$  means 1 is the only common factor for  $n$  and  $m$ .

(ii) We have the following set inclusion

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

$$\text{Also } \mathbb{I} \subset \mathbb{R} \text{ and } \mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

(iii) Note also note that the structure of the real line is such that unlike the set of integers the rational numbers and irrational numbers are everywhere. Thus in every interval (no matter how small) we can find there rational numbers and irrational numbers. We also say that between any two rational numbers (no matter how close they are to each other) there is an irrational number. This is



the situation that best relates to the story of siamese twins above. Thus the two largest subsets of  $\mathbb{Q}$  namely  $\mathbb{S}$  and  $\mathbb{S}^C$  behave like siamese twins such that we cannot separate them on the real line.

### Example 1a.1

We show that the number  $\sqrt{2}$  is irrational. We do this by contradiction. Suppose to the contrary that  $\sqrt{2}$  is rational. Then we can write

$$\sqrt{2} = \frac{n}{m}, \quad n, m \in \mathbb{Z}, m \neq 0, (n, m) = 1$$

$$\therefore \frac{n^2}{m^2} = 2$$

$$n^2 = 2m^2 \quad \text{is even}$$

$$\therefore n \text{ is even}$$

$$\text{i.e. } n = 2c \text{ for } c \in \mathbb{Z}$$

$$n^2 = (2c)^2 = 2m^2$$

$$4c^2 = 2m^2$$

$$2m^2 = 4c^2$$

$$m^2 = 2c^2 \text{ is even}$$

$$\therefore m \text{ is even}$$

But we cannot have both  $n$  and  $m$  even since  $(n, m) = 1$ . This is a contradiction.

Hence  $\sqrt{2}$  is irrational.

### Remark

The following example shows that from one known irrational number we can deduce many other irrational numbers. Here we use the fact that rationals are closed under both addition and multiplication.



### Example 1a.2

Let  $t$  be an irrational number. Show that

$$\frac{t+1}{t-1} \text{ is also irrational.}$$

### Solution

Let  $s = \frac{t+1}{t-1}$  and assume to the contrary that  $s$  is rational. Then we have

$$s(t-1) = t+1$$

$$st - s = t + 1$$

$$st - t = 1 + s$$

$$t(s-1) = s+1$$

$$t = \frac{s+1}{s-1}$$

Since rationals are closed under multiplication and addition we have that:

$$\frac{s+t}{s-1} \text{ is also rational.}$$

But  $t$  is irrational. This is a contradiction. Hence  $\frac{t+1}{t-1}$  is also irrational.

### Exercise 1a.3

(i) Given that  $a$  and  $b$  are rational numbers and  $s$  is irrational show that  $a - bs$  is also irrational.

(ii) Given that for  $p$  prime  $\sqrt{p}$  is irrational show that:

$$\frac{\sqrt{3}+1}{\sqrt{3}-1} \text{ is irrational.}$$





#### Example 1a.4

Let  $E \subseteq \mathbb{R}$  be given by  $E = \{x \in \mathbb{R} : 1 < x \leq 5\}$ . Then  $E$  is bounded below and above. In this case  $\inf E = 1$  and  $\sup E = 5$ .

Note that if  $\inf E \in E$  then  $\inf E$  is called minimum element of  $E$  written  $\min E$ . Also if  $\sup E \in E$  then  $\sup E$  is called the maximum element of  $E$  written  $\max E$ .

In the example above  $\inf E = 1 \notin E$  and thus  $E$  has no minimum element. But  $\sup E = 5 \in E$  and thus  $E$  has the maximum element which is 5.

#### Exercise 1a.5

- (i) Let  $E$  be a bounded set of real numbers such that  $a = \inf E$  and  $b = \max E$ . Give the definitions of  $a$  and  $b$ . Also give an example of a set where both  $a$  and  $b$  exist.

(ii) Let  $S = \left\{ \frac{1}{n} ; n \in \mathbb{J}^+ \right\}$

Thus  $S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$ .

State  $\sup S$ ,  $\inf S$ ,  $\max S$  and  $\min S$ .

#### 1a.6. Readings

1. Read the following sections from Elias Zakon, Basic Concepts in Maths, Trillia Press.

Part 1, chapters 8 and 9.

Part 2, chapters 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.

Complete and check all of the exercises in these sections.

2. Read the Wikipedia reference: Reimann Integrals.

For updates and links search for 'Reimann Integrals' at [www.wikipedia.org](http://www.wikipedia.org)



## Analysis 1 Module: Unit 1b

**Title:** The Riemann Integral

### Specific Objectives

At the end of this activity the learner will be able to:

- Define the Riemann integral
- Give examples of Riemann integrals defined on the set  $\mathbb{R}$
- State some properties of a Riemann integral
- Verify or prove some results on Riemann integrals.

### Summary

In this activity we formulate the Riemann integral which depends explicitly on the order structure of the real line. Accordingly we begin by discussing the concept of a partition of an interval and show that formulation of the Riemann integral is essentially one of the methods of estimating area under a given curve. Indeed it is the most accurate method of finding area bounded by a given curve. We thus define the relevant concepts like lower and upper sums that lead to lower and upper Riemann integrals respectively before we derive the actual Riemann integral.

We will also discuss briefly properties and classes of real valued functions which are Riemann integrable and give at least one example of a function which is not Riemann integrable. The extension of this theory to complex and vector valued functions on intervals and over sets other than intervals is subject matter of higher courses that can be discussed at graduate level.

### Key Words and Concepts

- **Partition of an interval:** Let  $[a, b]$  be an interval in  $\mathbb{R}$  and let  $x_0, x_1, \dots, x_n$  be such that  $a = x_0 < x_1 < x_2 < \dots < x_n = b$ . Then the set  $P = \{x_0, x_1, \dots, x_n\}$  is called a partition of  $[a, b]$  and the class of all partitions of  $[a, b]$  is denoted by  $P[a, b]$ .
- **Mesh of a partition:** let  $P$  be a partition of  $[a, b]$ . Then the number denoted by  $\mu(P)$  given by



$$\mu(p) = \max_{1 \leq i \leq n} |x_i - x_{i-1}|$$

where  $[x_{i-1}, x_i]$  is a sub interval of  $[a, b]$  is called the mesh or norm of  $P$ .

- **Finer Partition:** Let  $P_1$  and  $P_2$  be any two partitions of  $[a, b]$ . Then  $P_1$  is said to be finer or equivalently that  $P_2$  is coarser than  $P_1$  if every point of  $P_2$  is a point of  $P_1$ .
- **Lower and Upper Riemann Sums:** Let  $f$  be a bounded real-valued function on  $[a, b]$  and  $P$  be a partition of  $[a, b]$ .

$$\text{Let } \Delta x_i = x_i - x_{i-1} \quad \forall i=1, \dots, n$$

$$\text{Let } M = \text{lub} \left\{ f(x) : x \in [a, b] \right\}$$

$$m = \text{glb} \left\{ f(x) : x \in [a, b] \right\}$$

$$M_i = \text{lub} \left\{ f(x) : x \in [x_{i-1}, x_i] \right\}$$

$$m_i = \text{glb} \left\{ f(x) : x \in [x_{i-1}, x_i] \right\}$$

Clearly  $m \leq m_i \leq M_i \leq M \quad \forall i=1, \dots, n$ . Denote the sums

$$\sum_{i=1}^n M_i \Delta x_i \quad \text{and} \quad \sum_{i=1}^n m_i \Delta x_i \quad \text{by} \quad U(p, f) = \sum_{i=1}^n M_i \Delta x_i \quad \text{and}$$

$$L(p, f) = \sum_{i=1}^n m_i \Delta x_i \quad \text{respectively. Then } U(p, f) \quad \text{and} \quad L(p, f) \quad \text{are}$$

called the upper and lower Riemann sums respectively. Clearly

$$L(p, f) \leq U(p, f) \quad \forall p \in P [a, b].$$

- **Lower and Upper Riemann integrals:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Then the lower Riemann integral of  $f$  is given by

$$\int_a^b f(x) dx = \text{lub} \left\{ L(p, f) : p \in P [a, b] \right\}$$



and the upper Riemann integral of  $f$  is given by

$$\int_a^b \overline{f}(x) dx = \text{glb} \left\{ U(p, f) : p \in P[a, b] \right\}$$

- **Riemann integral of a function:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded real-valued function. Then  $f$  is said to be Riemann integrable if

$$\int_a^b \underline{f}(x) dx = \int_a^b \overline{f}(x) dx = \int_a^b f(x) dx.$$

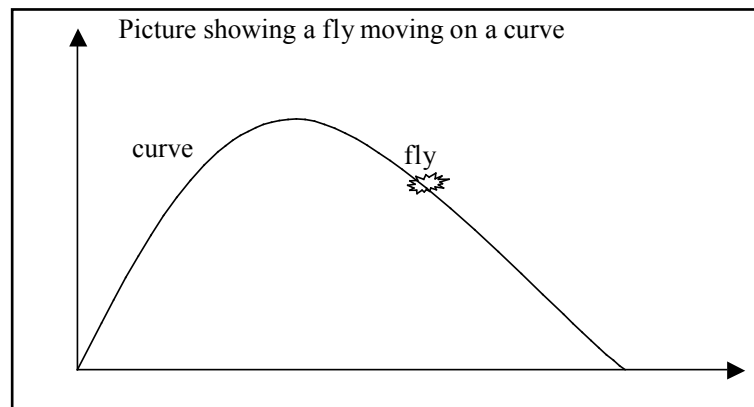
In this case the common value  $\int_a^b f(x) dx$  is called the Riemann integral of the function  $f$ .

### 1b Introduction The Story of Fred the Mathematical Fly

It is a common practice in our African culture that people tell stories or cite legends by referring to either animals or even insects when the message carried in such stories refer to behaviour of human beings. To this end our current story of modern times concerns Fred the mathematical fly who one day was moving on a curve whose equation was known.

He gave an assignment to his friends who were watching him as he moves on the given curve as follows:

Describe for me the procedure with which I can find my speed at any point on this curve. Also tell me the procedure to find most accurately the area bounded by this curve with respect to the two axes.





### Question

What are your answers to the two questions posed by Fred the mathematical fly in the story above?

### 1b.1 Introduction

We note that in the story above the two questions posed by the mathematical fly are addressing both theories of differential and integral calculus. Indeed we will show by the way of stating a theorem that differentiation and integration are in a way inverse operations.

We also note that from the definitions of lower and upper Riemann sums that clearly the formulation of the Riemann integral is based on estimating area under a given curve. Note that as we make the partition finer the lower Riemann sum increases to the actual area and the upper Riemann sum decreases to the actual area. Thus by taking the least upper bound of lower sums and the greatest lower bound of the upper sums we get a common value that becomes the Riemann integral.

### Remark 1b.1

The example below shows that there are real-valued functions which are not Riemann integrable.

### Example 1b.2.

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let  $P$  be any partition of  $[a, b]$ . Then in any of the subintervals

$$[x_{i-1}, x_i] \quad \forall i = 1, \dots, n.$$

we have:

$$M_i = \text{lub} \left\{ f(x) : x \in [x_{i-1}, x_i] \right\} = 1$$

$$m_i = \text{glb} \left\{ f(x) : x \in [x_{i-1}, x_i] \right\} = 0$$



Thus

$$\begin{aligned}
 U(p, f) &= \sum_{i=1}^n M_i \Delta x_i \\
 &= 1 \sum_{i=1}^n \Delta x_i \\
 &= 1(b-a) \\
 &= b-a > 0
 \end{aligned}$$

and

$$\begin{aligned}
 L(p, f) &= \sum_{i=1}^n m_i \Delta x_i \\
 &= 0 \sum_{i=1}^n \Delta x_i \\
 &= 0
 \end{aligned}$$

$$\therefore \text{glb}\{U(p, f) : p \in \rho[a, b]\} = 1 = \int_a^{\bar{b}} f \, dx$$

$$\text{lub}\{L(p, f) : p \in \rho[a, b]\} = 0 = \int_{\underline{a}}^b f \, dx$$

$$\text{Thus } \int_{\underline{a}}^b f \, dx = 0 \neq 1 = \int_a^{\bar{b}} f \, dx$$

Hence  $f$  so defined above is not Riemann integrable.

### Remark 1b.3

We state without proofs some of the theorems that give properties and classes of Riemann integrable functions.

### Theorem 1b.4 (Criterion for Riemann integrability)

Let  $f$  be bounded on  $[a, b]$ . Then  $f$  is Riemann integrable iff for each

$$\varepsilon > 0 \exists p \in P[a, b]$$



such that

$$U(p, f) - L(p, f) < \varepsilon$$

### Theorem 1b.5

Let  $f$  be bounded on  $[a, b]$ . If  $f$  is monotonic on  $[a, b]$  then  $f$  is Riemann integrable.

### Theorem 1b.6

Let  $f$  be bounded on  $[a, b]$ . If  $f$  is continuous, then  $f$  is Riemann integrable.

### Example 1b.7

Consider the function  $f : \left(0, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$  given by  $f(x) = \sin^2 x$

We note that  $f$  is continuous on  $\left(0, \frac{\pi}{2}\right]$  and hence Riemann integrable on

$$\left(0, \frac{\pi}{2}\right].$$

Indeed

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^2 x \, dx &= \int_0^{\frac{\pi}{2}} \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\ &= \left( \frac{1}{2} x - \frac{1}{4} \sin 2x \right) \Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4} - 0 = \frac{\pi}{4} \end{aligned}$$



### Theorem 1b.8

Let  $f$  and  $g$  be Riemann integrable functions defined on  $[a, b]$ , and  $c$  be a constant. Then we have:

- (i)  $f + g$  is Riemann integrable
- (ii)  $cf$  is Riemann integrable.

### Theorem 1b.9

Let  $f$  be bounded on  $[a, b]$ . If  $f$  is Riemann integrable then  $|f|$  is also Riemann integrable moreover,

$$\left| \int f \, dx \right| \leq \int |f| \, dx.$$

### Remark 1b.10

Note that we can also define the Riemann integral as a limit in the following way.

Let  $f$  be bounded on  $[a, b]$  and  $p$  be a partition of  $[a, b]$ . Let  $S(p, f)$  denote an arbitrary Riemann sum which may be either upper or lower. Then we have:

$$\int_0^{\pi} f \, dx = \lim_{\mu(p) \rightarrow 0} S(p, f)$$

This is because the finer the partition  $p$  is the closer we get to the actual integral and also the mesh  $\mu(p) \rightarrow 0$ .

If the partition  $p$  is chosen to have  $n$  subintervals so that  $\mu(p)$  can be expressed in terms of  $n$  then we have:

$$\int_a^b f \, dx = \lim_{n \rightarrow \infty} S(p, f).$$





**Remark 1b.11**

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be given by  $f(x) = x$ .

Let  $p$  be a partition of  $[0, 1]$  such that:

$$p = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{k}{n}, \dots, \frac{n}{n} = 1 \right\}$$

Consider the  $k^{\text{th}}$  subinterval of  $p$  i.e.  $[x_{k-1}, x_k]$ . Thus we have

$$\begin{aligned} M_k &= \text{lub} \left\{ f(x) : x \in [x_{k-1}, x_k] \right\} \\ &= f(x_k) = f\left(\frac{k}{n}\right) = \frac{k}{n}. \end{aligned}$$

$$\text{and } \Delta x_k = x_k - x_{k-1} = \frac{k}{n} - \frac{k-1}{n} = \frac{1}{n}$$

Hence

$$\begin{aligned} S(p, f) &= \sum_{k=1}^n M_k \Delta x_k \\ &= \sum_{k=1}^n \frac{k}{n} \cdot \frac{1}{n} \\ &= \frac{1}{n^2} \sum_{k=1}^n k \end{aligned}$$

$$\therefore \int_0^1 f \, dx = \int_0^1 x \, dx = \lim_{n \rightarrow \infty} \left( \frac{1}{n^2} \sum_{k=1}^n k \right)$$

**Exercise 1b.12**

(i) Show that the function defined by

$$f(x) = \begin{cases} 2 & \text{if } x \text{ is rational} \\ -2 & \text{if } x \text{ is irrational} \end{cases}$$

is not Riemann integrable.

(ii) Consider the function  $f : [0, a] \rightarrow \mathbb{R}$  given by  $f(x) = x^3$  where  $a$  is a constant. Show that

$$\int_0^a x^3 dx = \lim_{n \rightarrow \infty} \left( \frac{a^4}{n^4} \sum_{k=1}^n k^3 \right)$$

(iii) Given  $\frac{\pi}{n} \sum_{k=1}^n \sin \frac{k}{n} \pi$  as a Riemann sum, use this fact to evaluate

$$\lim_{n \rightarrow \infty} \left( \frac{\pi}{n} \sum_{k=1}^n \sin \frac{k}{n} \pi \right)$$

**Readings**

1. Theory of functions of a real variable by - Shilomo Stenberg (2005) pp 144
2. Read the Wikipedia reference: Riemann Integrals.

For updates and links search for 'Riemann Integrals' at [www.wikipedia.org](http://www.wikipedia.org)



## Analysis I Module: Unit 2

**Title:** On the Algebraic Structure with arithmetic operations in a space

### Specific Objectives

At the end of this activity the learner will be able to:

- Give definition of a vector space
- Identify the arithmetic operations among vectors
- Define a vector function on a vector space
- Determine differentiability of a vector function

### Summary

The richness of analysis in mathematics depends mainly on the structure of the spaces we deal with. As we have already stated in our first activity in Unit 1, the classical analysis started with the space  $\mathbb{R}$  in which the absolute value mapping gives it the geometric structure. However on the  $\mathbb{R}$  if we restrict ourselves to arithmetic operations like  $+$  (addition) and scalar product ( $\cdot$ ) then this will constitute an algebraic structure on  $\mathbb{R}$ . In this activity we consider a general vector space (see definition in the next section) and investigate how the structure gives rise to the definition of some concepts in the space. Indeed on the degree of smoothness e.g. differentiability of a function defined on a vector space.

### Key Words and Concepts

**Vector space:** This is a non-empty set of  $V$  on which we have two arithmetic operations namely  $\langle x, y \rangle \rightarrow x + y$  from  $V \times V$  to  $V$  called addition and  $\langle \alpha, x \rangle \rightarrow \alpha x$  from  $\mathbb{R} \times V$  to  $V$  called multiplications by scalors such that the following are satisfied.

$$A1 \quad x + (y + z) = (x + y) + z \quad \forall x, y, z, e$$

$$A2 \quad x + y = y + x \quad \forall x, y, \in V$$

$$A3 \quad \exists \text{ an element } o \in V \text{ such that } o + x = x \quad \forall x \in V$$

$$A4 \quad \text{For every } x \in V, \exists y \in V \text{ such that } x + y = 0$$

$$S1 \quad (\alpha \beta)x = \alpha(\beta x) \quad \forall \alpha, \beta \in \mathbb{R}, \forall x \in V$$

$$S2 \quad (\alpha + \beta)x = \alpha x + \beta x \quad \forall \alpha, \beta \in \mathbb{R}, \forall x \in V$$



$$S3 \quad \alpha(x + y) = \alpha x + \alpha y \quad \forall \alpha \in \mathbb{R}, \forall x, y \in \mathbb{V}$$

$$S4 \quad 1x = x \quad \forall x \in \mathbb{V}$$

Here A stands for property on operation of addition and S stands for property on operation of a scalar multiplication.

**Vector valued function:** This is a mapping  $r: \mathbb{R}^n$  that assigns a vector to each value of a parameter  $t$  thus:

$$r(t) = r_1(t)e_1 + \dots + r_n(t)e_n$$

e.g. position of an object in space as a function of time.

**Continuity:** A vector valued function is continuous at a point  $t = t_0$  if

$$\lim_{t \rightarrow t_0} r(t) = r(t_0)$$

This occurs iff the component functions are continuous.

**Differentiability:** A vector-valued function is differentiable if

$$\left( \frac{dr}{dt} \right) = \lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$$

exists.

This occurs if the component functions are differentiable.

**Gradient:** For a differential scalar field  $u(x)$ , the gradient of  $u$  is given by

$$\nabla u = \frac{\partial u}{\partial x_1} e_1 + \dots + \frac{\partial u}{\partial x_n} e_n$$



If the scalar field is  $\mathbb{R}^3$  then we have;

$$\nabla u = \frac{\partial u}{\partial x} \underline{i} + \frac{\partial u}{\partial y} \underline{j} + \frac{\partial u}{\partial z} \underline{k}$$

**Divergence of a vector Field:** for a vector field.

$$F(x, y, z) = M \underline{i} + N \underline{j} + P \underline{k},$$

The divergence of F denoted by  $\text{div } F$  is given by

$$\text{Div } F = \nabla \cdot F(x, y, z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z}$$

**Curl of a vector field:** For a vector field

$$F(x, y, z) = M \underline{i} + N \underline{j} + P \underline{k}$$

Curl of F denoted by  $\text{curl } F(x, y, z)$

Is given by  $\text{curl } F(x, y, z) = \nabla \times F(x, y, z)$

$$= \left( \frac{\partial p}{\partial y} - \frac{\partial n}{\partial z} \right) \underline{i} - \left( \frac{\partial p}{\partial x} - \frac{\partial m}{\partial z} \right) \underline{j} + \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) \underline{k}$$



## 2. On the Algebraic Structure with arithmetic operations in a space

### Story of Investigation in Gender Issues

In the year 2006 some time in October an N.G.O. dealing with gender issues in the city of Nairobi dispatched an agent to some premises of a business company. The company had just been started and the N.G.O. was eager to find out whether women also have some initiative to deal with the company. The task of this agent called Mwangi was to count the number of women in a random turnover on the premises within some specified time. While in the premises he stood at a point where people were passing to various directions. Some of them in groups so large that they could even push her out of the way and in the process she loses the count.

### Question

Could you suggest a better way or better point where Mwangi should have stood to have an accurate count?

### 2.0 Introduction

We note that in the story above rather than standing at a corner from where people are passing to all directions the agent should have used a point like the entrance or exit where movement is in only one particular direction. This is the main reason why we consider vector-valued functions because they create motion in a specific direction. Consequently the components of vector calculus like divergence and curl have applications in diverse fields like electromagnetism, fluid mechanism, etc.

We note that if the vector valued function  $\mathbf{r}(t)$ , represents the position of a moving ob-

ject then  $\frac{d\mathbf{r}}{dt}$  and  $\frac{d^2\mathbf{r}}{dt^2}$  are also vector valued functions representing velocity and acceleration respectively.

### Example 2.1

Consider the parametric curve:

$$\mathbf{r}(t) = \cos \frac{t}{2} \mathbf{i} + \sin \frac{t}{2} \mathbf{j}$$

Find velocity and acceleration vectors.

Find velocity and acceleration vectors.

**Solution**

$$\text{Velocity} = \frac{dr}{dt} = \frac{1}{2} \sin \frac{t}{2} \mathbf{i} + \frac{1}{2} \cos \frac{t}{2} \mathbf{j}$$

$$\text{Acceleration} = \frac{d^2v}{dt^2} = -\frac{1}{4} \cos \frac{t}{2} \mathbf{i} - \frac{1}{4} \sin \frac{t}{2} \mathbf{j}$$

**Exercise 2.2**

Let  $r(t)$  be the position of an object moving at constant speed. Show that the acceleration of the object is orthogonal to the velocity of the object.

**Example 2.3**

Given the vector field  $f(x, y, z) = x^3y^2z\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$  find the divergence at  $(2, 1, -1)$

**Solution**

$$\begin{aligned} \text{div}F(x, y, z) &= \frac{\partial}{\partial x}(x^3y^2z) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(x^2y) \\ &= 3x^2y^2z + 0 + 0 = 3x^2y^2z \end{aligned}$$

$$\begin{aligned} \text{At the point } (2, 1, -1) \text{div}F &= 3 \times 2^2 \times 1^2 \times (-1) \\ &= -12 \end{aligned}$$

**Exercise 2.4**

Find the divergence at the point  $(3, -2, 1)$  for the vector field

$$F(x, y, z) = x^2y^3z\mathbf{i} + xz^2\mathbf{j} + xy^2\mathbf{k}$$

**Example 2.5**

Find Curl  $F$  for the vector field given by

$$F(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2zy\mathbf{k}$$

**Solution**

$$\text{Curl } F(x, y, z) = \nabla \times F(x, y, z)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 + z^2 & 2zy \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + z^2 & 2xy \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xy & 2zy \end{vmatrix} \mathbf{j} +$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xy & x^2 + z^2 \end{vmatrix} \mathbf{k}$$

$$= (2z - 2z) \mathbf{i} - (0 - 0) \mathbf{j} + (2x - 2x) \mathbf{k}$$

$$= 0$$



**Exercise 2.6**

a) Given  $F(x, y, z) = x^2 + y^2 - z$

Find  $\nabla F$  at the point  $(1, 1, 2)$

b) Find the divergence at the point  $(-3, 2, 1)$  for the vector field

$$F(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2zy\mathbf{k}$$

c) Find the curl at the point  $(-1, 2, 1)$  for the vector field.

$$F(x, y, z) = x^3y^2z\mathbf{i} + x^2y\mathbf{k}$$

**Readings**

1. Larson Hostetler, Edwards, Calculus with analytic geometry, Houghton Mifflin Company New York 1988.
2. Divergence and Curl (Wikipedia)



## Analysis 1: Unit 3

**Title:** Calculus of functions of complex variables

### Specific Objectives

At the end of this activity the learner will be able to:

- Define a function of a complex variable;
- Give examples of functions of complex variables;
- Evaluate limits involving functions of complex variables;
- Determine continuity of a function of a complex variable;
- Determine differentiability of a function of a complex variable.

### Summary

We first note that from the Calculus Module the so called degree of smoothness of a function of a real-variable starts with definition where we have to state the domain and range. We then examine the existence of limit of such a function at some specified point. We also investigate continuity at a given point or on the whole domain followed by differentiability of the function at a point or on the whole domain in this case we have the following implications. Thus

Differentiability  $\Rightarrow$  Continuity  $\Rightarrow$  Existence of limit.

In this activity we go through the same process for the case of a function of a complex variable. However, it may be of great importance to note that a function  $f(z)$  of a complex variable  $z$  may also be viewed as a function of two real variables  $x$  and  $y$  since  $z = x + iy$  where  $x$  and  $y$  are real variables.

### Key Words and Concept

- **Function of a complex variable:** this is a function whose domain is the set  $\mathcal{C}$  of complex number field and range is either  $\mathcal{C}$  or  $\mathbb{R}$  for example  $f(z) = Z^2$  where  $z = x + iy$
- **Existence of limit:** for a function  $f : \mathcal{C} \rightarrow \mathcal{C}$  limit of  $f_2$  is said to exist at say  $z_0$  if given  $\Sigma > 0 \exists \delta > 0$  such that  $0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \Sigma$
- **Continuity:** For a function  $f : \mathcal{C} \rightarrow \mathcal{C}$  continuity of  $f$  at  $z_0$  means for each  $\Sigma > 0 \exists \delta > 0$  such that:  $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \Sigma$



- **Differentiability:** A function  $f : \mathcal{C} \rightarrow \mathcal{C}$  is said to be differentiable at  $z_0$  if

limit  $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  exists. In this case the limit above is denoted by

$$f'(z_0) = \frac{f(z) - f(z_0)}{z - z_0}$$

- **Analytic function:** A function  $f : \mathcal{C} \rightarrow \mathcal{C}$  is said to be analytic at a point  $z_0$  if  $f$  is differentiable in some neighborhood of the point  $z_0$ . Thus  $f'(z)$  exists in some neighborhood of  $z_0$ .
- **Entire function:** A function  $f : \mathcal{C} \rightarrow \mathcal{C}$  is said to be entire if  $f$  is analytic at every point in its domain.

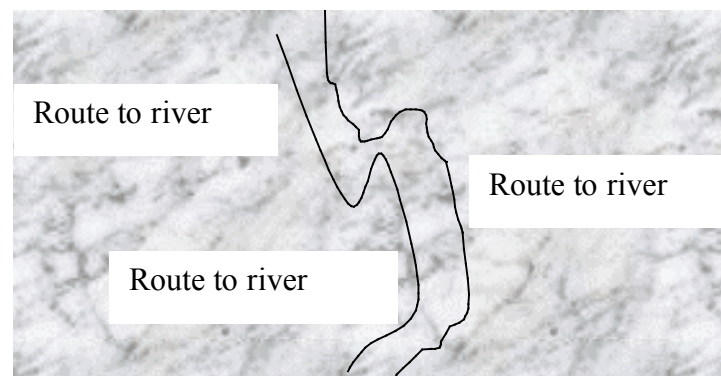
### 3 Calculus of functions of complex variables

#### Story of a river in the bush

River Sosio can be located in one of the districts in Kenya called Bungoma. It is a long river which runs through several villages most of them with bushes along the river. Some years back the local administration wanted to put bridges at various points on the river to facilitate human transactions across it. It was difficult to determine at which points to put these bridges because it was observed that in most areas we could have routes leading to the river on one side but directly opposite it was so bushy that no route reaches the river.

It was also observed that in some areas we could have routes which diminish before reaching the river banks.

*Picture showing long Winding River with routes on either side but not directly opposite each other*





### Question

Can you deduce a condition to be satisfied by the routes on either side of the river so that we determine where to put a bridge?

### 3.0 Introduction

Recall that in the module of Calculus we came across the concept of continuity of a function of a real variable in which we saw that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at say  $x_0$  if the following three conditions are satisfied:

- i)  $f$  is defined at  $x_0$
- ii) Limit  $f(x)$  exists  $x \rightarrow x_0$
- iii) Limit  $f(x) = f(x_0)$   $x \rightarrow x_0$

This situation prevails even in the case of functions of complex variables. Although in our section of key words and concepts we have used E, S definition of continuity we note that this is just an equivalent statement which ensures that the three conditions above are satisfied. The story of the river above says everything about this situation. Indeed we must have routes leading to the river on either side and reaching the river banks directly opposite each other so that a bridge provides continuity of movement across the river and on either side of it. For routes that do not reach the river it means that limit does not even exist for them at the river and therefore, we cannot leave continuity of movement across the river.

### Example 3.1

$$z^2 = z_0^2$$

### Remark

The example above shows that in some cases particularly when we are dealing with continuous functions we can evaluate limits in the same way as functions of real variables. However, with respect to complex variables a function of a complex variable can be regarded as a function of two real variables. This leads to partial evaluation of limits and also when it comes to differentiability we deal with partial derivatives. We state some theorems below to help this theory to this end.



### Theorem 3.2

Let  $z = x + iy$  and  $f$  be a function of  $z$ . Then we can express  $f$  into its real and imaginary parts as:

$$f(z) = u(x, y) + iv(x, y) \text{ For a fixed } z_0 = x_0 + iy_0 \text{ we have } \lim_{x \rightarrow x_0} u(x, y_0) = a =$$

$$\lim_{y \rightarrow y_0} u(x_0, y)$$

$$\text{And } \lim_{x \rightarrow x_0} v(x, y_0) = b = \lim_{x \rightarrow x_0} v(x_0, y)$$

### Remark

Note that the Cauchy-Riemann conditions are only necessary but not sufficient for the derivative to exist. Thus for the converse of the theorem above we require the following result.

### Theorem 3.3 (Cauchy-Riemann condition)

If  $f(z_0)$  exists then

$$\frac{\partial u}{\partial x} \Big|_{z_0} = \left( \frac{\partial v}{\partial y} \right)_{z_0} \text{ and}$$

$$\left( \frac{\partial v}{\partial x} \right)_{z_0} = - \left( \frac{\partial u}{\partial y} \right)_{z_0}$$

$$\text{Also } f'(z_0) = \left( \frac{\partial u}{\partial x} \right)_{z_0}$$

$$= \left( \frac{\partial v}{\partial y} \right)_{z_0} - i \left( \frac{\partial u}{\partial y} \right)_{z_0}$$

### Remark

Not that the Cauchy Riemann condition are only necessary but not sufficient for the derivative to exist. Thus the converse of the theory above we require the following results.



### Therefore

Suppose  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$  exist in a neighborhood of a point  $(x_0, y_0)$  and are all continuous at  $(x_0, y_0)$ . Then if Cauchy –Riemann condition are satisfied at  $Z_0$ ,  $f(Z_0)$  exists.

### Example 3.4

We now show using partial limits that for  $z_0 = x_0 + iy_0$   $\lim_{z \rightarrow z_0} z^2 = z_0^2$

$$\begin{aligned} \text{Thus } f(z) &= z^2 \\ &= (x + iy)^2 \\ &= (x^2 - y^2) + 2ixy \\ \therefore u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy \end{aligned}$$

$$\therefore \lim_{x \rightarrow x_0} u(x, y_0) = \lim_{x \rightarrow x_0} (x^2 - y_0^2) = x_0^2 - y_0^2$$

$$\lim_{y \rightarrow y_0} u(x_0, y) = \lim_{y \rightarrow y_0} (x_0^2 - y^2) = x_0^2 - y_0^2$$

$$\lim_{x \rightarrow x_0} v(x, y_0) = \lim_{x \rightarrow x_0} (2xy_0) = 2x_0y_0$$

$$\lim_{y \rightarrow y_0} v(x_0, y) = \lim_{y \rightarrow y_0} (2x_0y) = 2x_0y_0$$

$$\therefore \lim_{z \rightarrow z_0} z^2 = (x_0^2 - y_0^2) + i(2x_0y_0) = z_0^2$$



### Exercise 3.5

Show that:  $\lim_{z \rightarrow 0} \frac{\bar{z}}{z}$  does not exist.

### Example 3.6

Given  $f(z) = z^2$  show that  $f'(z_0) = 2z_0$ .

$$\begin{aligned} \text{Now } f(z) &= z^2 = (x + iy)^2 \\ &= x^2 - y^2 + i(2xy) \\ \therefore u(x, y) &= x^2 - y^2 \text{ and} \\ v(x, y) &= 2xy \end{aligned}$$

$$\text{i.e. } \left. \frac{\partial u}{\partial x} \right|_{(x_0, y_0)} = 2x_0$$

$$\left. \frac{\partial u}{\partial y} \right|_{(x_0, y_0)} = -2y_0$$

$$\left. \frac{\partial v}{\partial x} \right|_{(x_0, y_0)} = 2y_0$$

$$\left. \frac{\partial v}{\partial y} \right|_{(x_0, y_0)} = 2x_0$$

The Cauchy-Riemann conditions are satisfied. Thus  $f'(z_0)$  exists and:

$$\begin{aligned} f'(z_0) &= \left. \frac{\partial v}{\partial x} \right|_{(x_0, y_0)} + i \left. \frac{\partial v}{\partial y} \right|_{(x_0, y_0)} \\ &= 2x_0 + i2y_0 \\ &= 2z_0 \end{aligned}$$

**Exercise 3.7**

a) Let  $f(z) = |z|^2$  show that  $f'(z)$  does not exist for  $z \neq 0$

b) Let

$$f(z) = \begin{cases} \frac{\overline{z^2}}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Show that Cauchy-Riemann conditions are satisfied but  $f'(0)$  does not exist.

c) Given  $f(z) = e^z$  show that:

i)  $f(z)$  is analytic at any point  $z = z_0$

ii)  $f(z)$  is entire

d) Let  $f(z) = e^{\overline{z}}$  show that  $f(z)$  is not analytic anywhere.

**Readings**

1. Students should read the following chapters. Complete and check the exercises given.

Functions of a Complex Variable Chapters 8 to 13 (p.p. 360 to 771) (numbered p.388 to 799 in the PDF document) in Introduction to Methods of Applied Mathematics Sean Mauch <http://www.its.caltech.edu/~sean>





## X. Compiled list of all Key Concepts (Glossary)

### Key Concepts

**Rational number:** This is a number that can be written in the form  $\frac{n}{m}$  where  $n$  and  $m$  are integers with  $m \neq 0$  and 1 as the only common factor for  $n$  and  $m$ .

**Irrational number:** This is a number that cannot be written in the form  $\frac{n}{m}$ , where  $n$  and  $m$  are integers with  $m \neq 0$ . Thus, we have that the set of rational numbers is  $\mathbb{Q}$  and the set of irrational numbers  $\mathbb{Q}^C$  leading to the relation

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^C$$

i.e. Every real number is either rational or irrational

**Bounded below:** Any set  $E$  of real numbers is said to be bounded below if there is a real number  $q$  such that  $q \leq x \quad \forall x \in E$ . In this case  $q$  is called a lower bound for  $E$ .

**Bounded above:** Any set  $E$  of real numbers is said to be bounded above if there is a real number  $p$  such that  $p \geq x \quad \forall x \in E$ . In this case  $p$  is called an upper bound for  $E$ .

**Infimum:** For any set  $E$  of real numbers, the largest of all real numbers  $q$  such that  $q \leq x \quad \forall x \in E$ . is called infimum of  $E$  written  $\text{Inf } E$  or is called greatest lower bound written  $\text{glb } E$ .

**Supremum:** For any set  $E$  of real numbers the smallest of all real numbers  $p$  such that  $p \geq x \quad \forall x \in E$  is called the supremum of  $E$  written  $\text{sup } E$  or is called the least upper bound of  $E$  written  $\text{lub } E$ .

**Completeness axiom:** This states as follows: Every non empty set  $E$  of real numbers bounded above has supremum and if bounded below has infimum.



**Partition of an interval:** Let  $[a, b]$  be an interval in  $\mathbb{R}$  and let  $x_0, x_1, \dots, x_n$  be such that  $a = x_0 < x_1 < x_2 < \dots < x_n = b$ . Then the set  $P = \{x_0, x_1, \dots, x_n\}$  is called a partition of  $[a, b]$  and the class of all partitions of  $[a, b]$  is denoted by  $\mathcal{P}[a, b]$ .

**Mesh of a partition:** let  $P$  be a partition of  $[a, b]$ . Then the number denoted by  $\mu(p)$  given by

$$\mu(p) = \max_{1 \leq i \leq n} |x_i - x_{i-1}|$$

where  $[x_{i-1}, x_i]$  is a sub interval of  $[a, b]$  is called the mesh or norm of  $P$ .

**Finer Partition:** Let  $P_1$  and  $P_2$  be any two partitions of  $[a, b]$ . Then  $P_1$  is said to be finer or equivalently that  $P_2$  is coarser than  $P_1$  if every point of  $P_2$  is a point of  $P_1$ .

**Lower and Upper Riemann Sums:** Let  $f$  be a bounded real-valued function on  $[a, b]$  and  $P$  be a partition of  $[a, b]$ .

$$\text{Let } \Delta x_i = x_i - x_{i-1} \quad \forall i = 1, \dots, n$$

$$\text{Let } M = \text{lub} \left\{ f(x) : x \in [a, b] \right\}$$

$$m = \text{glb} \left\{ f(x) : x \in [a, b] \right\}$$

$$M_i = \text{lub} \left\{ f(x) : x \in [x_{i-1}, x_i] \right\}$$

$$m_i = \text{glb} \left\{ f(x) : x \in [x_{i-1}, x_i] \right\}$$

Clearly  $m \leq m_i \leq M_i \leq M \quad \forall i = 1, \dots, n$ . Denote the sums

$$\sum_{i=1}^n M_i \Delta x_i$$

$$\text{and } \sum_{i=1}^n m_i \Delta x_i \text{ by } U(p, f) = \sum_{i=1}^n M_i \Delta x_i$$



and  $L(p, f) = \sum_{i=1}^n m_i \Delta x_i$  respectively. Then  $U(p, f)$  and  $L(p, f)$  are called the upper and lower Riemann sums respectively. Clearly

$$L(p, f) \leq U(p, f) \quad \forall p \in \mathcal{P}[a, b].$$

**Lower and Upper Riemann integrals:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Then the lower Riemann integral of  $f$  is given by

$$\int_a^b f(x) dx = \text{lub} \left\{ L(p, f) : p \in \mathcal{P}[a, b] \right\}$$

and the upper Riemann integral of  $f$  is given by

$$\int_a^b f(x) dx = \text{glb} \left\{ U(p, f) : p \in \mathcal{P}[a, b] \right\}$$

**Riemann integral of a function:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded real-valued function. Then  $f$  is said to be Riemann integrable if

$$\int_a^b f(x) dx = \int_a^b f(x) dx = \int_a^b f(x) dx.$$

In this case the common value  $\int_a^b f(x) dx$  is called the Riemann integral of the function  $f$ .



**Vector space:** This is a non-empty set of  $V$  on which we have two arithmetic operations namely  $\langle x, y \rangle \rightarrow x + y$  from  $V \times V$  to  $V$  called addition and  $\langle \alpha, x \rangle \rightarrow \alpha x$  from  $R \times V$  to  $V$  called multiplications by scalors such that the following are satisfied.

$$A1 \quad x + (y + z) = (x + y) + z \quad \forall x, y, z, e$$

$$A2 \quad x + y = y + x \quad \forall x, y, \in V$$

$$A3 \quad \exists \text{ an element } o \in V \text{ such that } o + x = x \quad \forall x \in V$$

$$A4 \quad \text{For every } x \in V, \exists y \in V \text{ such that } x + y = 0$$

$$S1 \quad (\alpha \beta)x = \alpha(\beta x) \quad \forall \alpha, \beta \in R, \forall x \in V$$

$$S2 \quad (\alpha + \beta)x = \alpha x + \beta x \quad \forall \alpha, \beta \in R, \forall x \in V$$

$$S3 \quad \alpha(x + y) = \alpha x + \alpha y \quad \forall \alpha \in R, \forall x, y \in V$$

$$S4 \quad 1x = x \quad \forall x \in V$$

Here A stands for property on operation of addition and S stands for property on operation of a scalar multiplication.

**Vector valued function:** This is a mapping  $r : R^n$  that assigns a vector to each value of a

Parameter  $t$  thus:

$$r(t) = r_1(t)e_1 + \dots + r_n(t)e_n$$

e.g. position of an object in space as a function of time.

**Continuity:** A vector valued function is continuous at a point  $t = t_0$  if

$$\lim_{t \rightarrow t_0} r(t) = r(t_0)$$

$$t \rightarrow t_0$$

This occurs iff the component functions are continuous.

**Differentiability:** A vector-valued function is differentiable if

$$\left( \frac{dr}{dt} \right) = \lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$$

exists.



This occurs if the component functions are differentiable.

**Gradient:** For a differential scalar field  $u(\mathbf{x})$ , the gradient of  $u$  is given by

$$\nabla u = \frac{\partial u}{\partial x_1} \mathbf{e}_1 + \dots + \frac{\partial u}{\partial x_n} \mathbf{e}_n$$

If the scalar field is  $\mathbb{R}^3$  then we have;

$$\nabla u = \frac{\partial u}{\partial x_1} \mathbf{i} + \frac{\partial u}{\partial x_2} \mathbf{j} + \frac{\partial u}{\partial x_3} \mathbf{k}$$

**Divergence of a vector Field:** for a vector field.

$$\mathbf{F}(x, y, z) = M \mathbf{i} + N \mathbf{j} + P \mathbf{k},$$

The divergence of  $\mathbf{F}$  denoted by  $\text{div } \mathbf{F}$  is given by

$$\text{Div } \mathbf{F} = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z}$$

Curl of a vector field: For a vector field

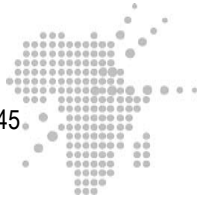
$$\mathbf{F}(x, y, z) = M \mathbf{i} + N \mathbf{j} + P \mathbf{k}$$

Curl of  $\mathbf{F}$  denoted by  $\text{curl } \mathbf{F}(x, y, z)$

Is given by  $\text{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z)$

$$= \left( \frac{\partial p}{\partial y} - \frac{\partial n}{\partial z} \right) \mathbf{i} - \left( \frac{\partial p}{\partial x} - \frac{\partial m}{\partial z} \right) \mathbf{j} + \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) \mathbf{k}$$

**Function of a complex variable:** this is a function whose domain is the set  $\mathcal{C}$  of complex number field and range is either  $\mathcal{C}$  or  $\mathbb{R}$  for example  $f(z) = Z^2$  where  $z = x + iy$



**Existence of limit:** for a function  $f : \mathcal{C} \rightarrow \mathcal{C}$  limit of  $f_2$  is said to exist at say  $z_0$  if given  $\Sigma > 0 \exists \delta > 0$  such that  $0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \Sigma$

**Continuity:** For a function  $f : \mathcal{C} \rightarrow \mathcal{C}$  continuity of  $f$  at  $z_0$  means for each  $\Sigma > 0 \exists \delta > 0$  such that:  $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \Sigma$

**Differentiability:** A function  $f : \mathcal{C} \rightarrow \mathcal{C}$  is said to be differentiable at  $z_0$

if limit  $z \rightarrow z_0 \frac{f(z) - f(z_0)}{z - z_0}$  exists. In this case the limit above is denoted by

$$f'(z_0) = \frac{f(z) - f(z_0)}{z - z_0}$$

**Analytic function:** A function  $f : \mathcal{C} \rightarrow \mathcal{C}$  is said to be analytic at a point  $z_0$  if  $f$  is differentiable in some neighborhood of the point  $z_0$ . Thus  $f'(z)$  exists in some neighborhood of  $z_0$

**Entire function:** A function  $f : \mathcal{C} \rightarrow \mathcal{C}$  is said to be entire if  $f$  is analytic at every point in its domain



## XI. Compiled list of Compulsory Readings

Elias Zakon, Basic Concepts in Maths, Trillia Press.

Part 1, chapters 8 and 9.

Part 2, chapters 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.

Wikipedia reference: Reimann Integrals.

For updates and links search for 'Reimann Integrals' at [www.wikipedia.org](http://www.wikipedia.org)

Functions of a Complex Variable Chapters 8 to 13 (p.p. 360 to 771) (numbered p.388 to 799 in the PDF document) in Introduction to Methods of Applied Mathematics Sean Mauch <http://www.its.caltech.edu/~sean>



## XII. Compiled List Of (Optional) Multimedia Resources

### Reading # 1: Wolfram MathWorld (visited 03.11.06)

**Complete reference :** <http://mathworld.wolfram.com>

**Abstract :** Wolfram MathWorld is a specialised on-line mathematical encyclopedia.

**Rationale:** It provides the most detailed references to any mathematical topic. Students should start by using the search facility for the module title. This will find a major article. At any point students should search for key words that they need to understand. The entry should be studied carefully and thoroughly.

### Reading # 2: Wikipedia (visited 03.11.06)

**Complete reference :** <http://en.wikipedia.org/wiki>

**Abstract :** Wikipedia is an on-line encyclopedia. It is written by its own readers. It is extremely up-to-date as entries are continually revised. Also, it has proved to be extremely accurate. The mathematics entries are very detailed.

**Rationale:** Students should use wikipedia in the same way as MathWorld. However, the entries may be shorter and a little easier to use in the first instance. They will, however, not be so detailed.

### Reading # 3: MacTutor History of Mathematics (visited 03.11.06)

**Complete reference :** <http://www-history.mcs.standrews.ac.uk/Indexes>

**Abstract :** The MacTutor Archive is the most comprehensive history of mathematics on the internet. The resources are organised by historical characters and by historical themes.

**Rationale:** Students should search the MacTutor archive for key words in the topics they are studying (or by the module title itself). It is important to get an overview of where the mathematics being studied fits in to the history of mathematics. When the student completes the course and is teaching high school mathematics, the characters in the history of mathematics will bring the subject to life for their students. Particularly, the role of women in the history of mathematics should be studied to help students understand the difficulties women have faced while still making an important contribution.. Equally, the role of the African continent should be studied to share with students in schools: notably the earliest number counting devices (e.g. the Ishango bone) and the role of Egyptian mathematics should be studied.





### **XIII. Synthesis of the Module**

We note that having gone through this module you should now be fully equipped with the knowledge of the concepts involved in the following contents:

In unit 1 the basic concepts are those of the structure of the real number system as constituted by both rational and irrational numbers leading to a discussion of bounded sets of real numbers. We also expect that you have had a good grasp of properties of functions defined on the set of real numbers. These properties include continuity, differentiability and integrability leading to the concept of the Riemann Integral.

In unit 3 the structure of a vector space is essential in the sense that it stresses the use of arithmetic operations together with vector operations before getting into the calculus of vector valued functions. In the calculus, we first looked at the differentiability in which the concepts of divergence and curl are prominent because of their applications in various fields. We then come to types of integration including double and triple integrals leading to theorems like Green's, Stoke's etc.

Finally in unit 3 we have gone through calculus of functions of complex variables in which concepts of continuity, differentiability and analyticity should now be clear. The integration of such functions using various techniques is well exposed before completing the unit with a coverage on types of power series such as Taylor's, McLaurin's etc.



## XIV. Summative Evaluation

### Final Assessment

Q1. Given that  $a$  and  $b$  are rational numbers, show that

$T = a - bs$  is irrational hence show that

$\frac{\sqrt{2} + 1}{\sqrt{2} - 1}$  is also irrational.

Q2. a) State the completeness axiom

b) Let  $E$  be a closed and bounded subset of  $\mathbb{R}$ . Show that  $\text{Sup } E$  and  $\text{Inf } E$  belong to  $E$ .

Q3. Let  $(x_n)$  be a sequence of real numbers. Show that if  $x_n \rightarrow x$  then  $|x_n| \rightarrow |x|$  is the converse true?

Justify your answer.

Q4. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = \frac{1}{x}, x > 0$

Show that in the  $\epsilon$ - $\delta$  definition of continuity of  $f$  at the point  $x_0 = 1$ , the values

If  $\epsilon = \frac{1}{2}$  and  $\delta = \frac{1}{3}$  will do.

Q5. Show that the function  $f$  defined by

$f(x) = 1$ , if  $x$  is irrational  
 $-1$ , if  $x$  is rational

is not Riemann integrable

Q6. Show that:  $\lim_{z \rightarrow 0} \frac{\bar{z}}{z}$  does not exist.



Q7. Give the definition of an analytic function of a complex variable. Show that  $f(z) = e^{\bar{z}}$  is not analytic anywhere.

Q8. By parameterising the complex number  $z$  evaluate:  $\int_C \frac{1}{z} dz$  where  $C$  is circle within centre origin and radius  $R$ . Evaluate the same integral if  $C$  is any other circle which does not include the origin in its interior nor boundary. State any results you use.

Q9. Let  $r(t)$  be the position of a particle moving at constant speed.

(a) Show that the acceleration of the particle is orthogonal to the velocity of the object.

(b) For the vector valued function  $F(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} - z\mathbf{k}$  find  $\nabla F$  at the point  $(2, -1, 1)$

Q10. It is known that a vector field  $F$  is called divergence free field if  $\text{div}F = 0$  show that the vector field  $F(x, y, z) = x^3y^2z\mathbf{i} + x^2z^2\mathbf{j} + x^2y^2z\mathbf{k}$  is not divergence free unless  $x = 0$  or  $y = 0$  or  $z = 0$

It is also known that a vector  $F$  is conservative if  $\text{Curl } F = 0$ . Show that the vector field

$F(x, y, z) = xzy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2zy\mathbf{k}$  is conservative



### Final assessment: Solutions

Q1. Assume  $t=a+bs$  is rational. Then we have that

$$bs = -(a - t)$$

$$s = -\frac{(a - t)}{b} = \frac{t - a}{b}$$

Since the set of rational numbers is closed under addition and multiplication  $\frac{t - a}{b}$ . But  $s$  is irrational. This is a contradiction. Hence it is irrational

$$\begin{aligned} \text{Now } \frac{\sqrt{2} + 1}{\sqrt{2} - 1} &= \frac{(\sqrt{2} + 1)(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)} \\ &= \frac{2 + \sqrt{2} + \sqrt{2} + 1}{2 + \sqrt{2} - \sqrt{2} - 1} \\ &= \frac{3 + 2\sqrt{2}}{1} = 3 + 2\sqrt{2} \end{aligned}$$

Since 3 and 2 are rational and  $\sqrt{2}$  is irrational we have by the result that  $3 + 2\sqrt{2}$

is irrational. Hence  $\frac{\sqrt{2} + 1}{\sqrt{2} - 1}$  is irrational



### Completeness axiom

Q2 a) Every set of real numbers bounded above has the least upper bound while every set of real numbers bounded below has the greatest lower bound.

b) Let  $b = \sup E$  and  $q = \inf E$ . Then for each  $\epsilon > 0$ ,  $b - \epsilon$  is not an upper bound for  $E$

This  $\exists b' \in E$  such that  $b - \epsilon < b' < b$ . This is equivalent to saying that every neighborhood  $N(b, \epsilon)$  of  $b$  has at least one point of  $E$ . Since  $E$  is closed  $b = \sup E \in E$  is closed  $b = \sup E \in E$ .

Similarly for each  $\epsilon > 0$   $q + \epsilon$  is not a lower bound of  $E$ . Thus  $\exists q' \in E$  such that  $q < q' < q + \epsilon$ . This implies  $q$  is a limit point of  $E$ . Now since  $E$  is closed  $q = \inf E \in E$ .

We first of all note that  $\|x_n - x\| < \epsilon \iff |x_n - x| < \epsilon \forall n > M$

Hence  $|x_n| \rightarrow |x|$

However the converse is not true in general as the following counter example shows.

### Counter example

Consider the sequence

$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$$

Then  $|x_n| \rightarrow 1$

But  $(x_n)$  does not converge in  $\mathbb{R}$  since it oscillates between  $-1$  and  $1$ .



Q4. Given  $f(x) = \frac{1}{x}, x > 0$  we show that  $f$  is continuous at  $x_0 = 1$ .

Given in particular that  $\epsilon = \frac{1}{2}$  we have.

$$\left| f(x) - f(1) < \epsilon \Rightarrow \left| \frac{1}{x} - 1 \right| < \frac{1}{2}$$

$$\Rightarrow \left| \frac{1-x}{x} \right| < \frac{1}{2}$$

$$\Rightarrow \left| -1 \frac{(x-1)}{1x1} \right| < \frac{1}{2}$$

$$\Rightarrow \left| \frac{|x-1|}{1x1} \right| < \frac{1}{2}$$

$$\Rightarrow |x-1| < \frac{|x|}{2}$$

$$\Rightarrow |x-1| < \frac{x}{2} \text{ for } x > 0$$

Now take  $\delta = \frac{x}{2}$ , then  $|x-1| < \delta \Rightarrow \left| \frac{1}{x} - 1 \right| < \frac{1}{2}$

But since  $x$  is any positive number in particular let  $x = \frac{2}{3}$ . Then we have

$$|x-1| < \delta = \frac{1}{3} \Rightarrow \left| \frac{1}{x} - 1 \right| < \frac{1}{2}$$



$$\text{Q5. Let } M_i = \text{lub} \left\{ f(x) : x \in (x_i, x_{i+1}) \right\} = 1$$

$$M_i = \text{glb} \left\{ f(x) : x \in (x_{i-1}, x_i) \right\} = 1$$

$$\text{Then } U(p, f) = \sum_{i=1}^n \Delta x_i (b-a) < \epsilon$$

$$L(p, f) = \sum_{i=1}^n m_i \Delta x_i = -1 \sum_{i=1}^n \Delta x_i$$

$$= -1 \left( \pi \frac{(b-a)}{b} \right) < \epsilon$$

$$\text{Thus } \int_a^b f(x) dx = -1(b-a) < \epsilon < \int_a^b f(x) dx = b-a$$

Hence  $f$  is not Riemann integrable.

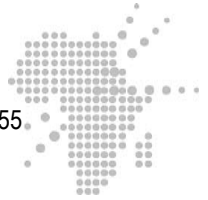
$$\text{Q6. Let } f(z) = \frac{\bar{z}}{z} = u + iv$$

$$= \frac{(x-iy)^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} + i \frac{-2xy}{x^2+y^2}$$

$$\therefore U(x, y) = \frac{x^2-y^2}{x^2+y^2}$$

$$V(x, y) = \frac{-2xy}{x^2+y^2}$$



$$\text{Now } \lim_{x \rightarrow 0} U(x, 0) = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1$$

$$\lim_{y \rightarrow 0} U(0, y) = \frac{y^2}{y^2} = -1$$

Hence the limit does not exist.

Q7. A function  $f(z)$  of a complex variable is said to be analytic at  $Z_0$  if  $f(z)$  is differentiable at every  $Z$  in some nbl of  $Z_0$ .

$$\text{Given } f(z) = e^z$$

$$\begin{aligned} \therefore f(z) &= e^{x-iy} \\ &= e^x (\cos y - i \sin y) \end{aligned}$$

$$\text{Let } U = e^x \cos y, \quad V = -e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \frac{\partial v}{\partial x} = -e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \quad \frac{\partial v}{\partial y} = -e^x \cos y$$

$$\therefore \frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$$

and

$$\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$$



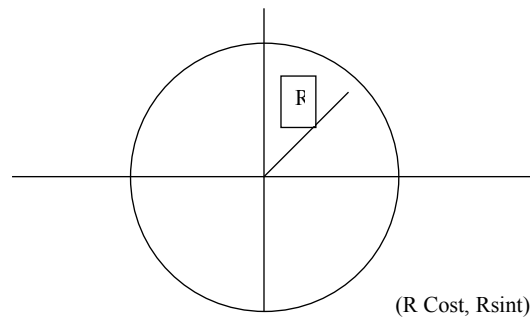


Therefore Cauchy Reimman conditions are not certified unless  $\cos y = 0 \Rightarrow y$

$$= + \frac{\pi + 2n\pi}{2}$$

But  $f(z)$  not differentiable in any nbl of a point on their line  $y = + \frac{\pi}{2} + 2n\pi$   
Hence  $f$  is not analytic anywhere.

Q8. We first note that the contour is a circle



$$\text{Thus } \begin{cases} x = R \cos t \\ y = R \sin t \end{cases} t \in [0, 2\pi]$$

$$\therefore Z = X + iy = R e^{it}$$

$$\therefore \int_0^{2\pi} \frac{1}{Z} dz = \int_0^{2\pi} \frac{1}{R e^{it}} i R e^{it} dt$$

$$= \int_0^{2\pi} i dt$$

it

Of Cauchy-Goursat which says that if  $f(z)$  is analytical and inside a simple closed contour  $c$ , then  $\int f(z) dz = 0$



9. a) If  $r(t)$  has constant speed then  $|r'(t)| = c$ . Since acceleration is orthogonal to the velocity we have:

$$r'(t) \cdot r'(t) = c^2$$

$$\text{i.e. } \frac{d}{dt} (r'(t) \cdot r'(t)) = 0$$

$$\text{i.e. } 2r'(t) \cdot r''(t) = 0$$

Differentiating gives  $r'(t) \cdot r''(t) + r''(t) \cdot r'(t) = 0$  i.e.  $r'(t) \cdot r''(t) = 0$  Hence acceleration is orthogonal to velocity.

- b) Given:  $F(x, y, z) = x^2i + 2y^2j - zk$   
 $\nabla F = 2xi + 4yj - k$   
 At the point  $(2, -1, 1)$  we have  $\nabla F = 4i - 4j - k$

$$F(x, y, z) = x^3y^2z + x^2yz + x^2yk$$

10. a) Given  $\text{div}F = \frac{\partial}{\partial x}(x^3y^2z) + \frac{\partial}{\partial y}(x^2y^2) + \frac{\partial}{\partial z}(x^2y)$   
 $= 3x^2y^2z + 2xy^2 + x^2y$   
 $= 3x^2y^2z \neq 0$  unless either  $x = 0$ ,  $y = 0$  or  $z = 0$ .

Hence  $F(x, y, z)$  is divergence free.

- b) Given  $F(x, y, z) = 2xyi + (x^2 + z^2)j + 2zyk$  we have:

$$\text{Curl } F(x, y, z) = \nabla \times F(x, y, z)$$

$i$	$j$	$k$
$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial z}$
$\frac{\partial}{\partial xy}$	$x^2 + z^2$	$\frac{\partial}{\partial zy}$
$=$	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial z}$
$x^2 + z^2$	$2zy$	$\frac{\partial}{\partial x}$
		$\frac{\partial}{\partial z}$
		$\frac{\partial}{\partial xy}$
		$\frac{\partial}{\partial zy}$



$$\begin{aligned} & + \frac{\partial}{\partial x} \frac{\partial}{\partial x} K \\ & = 2xy \quad x^2 + z^2 \\ & = (2z - 2z)i - (0 - 0)j + (2x + 2x)k \\ & = 0 - 0 + 0 = 0 \end{aligned}$$

Hence  $F(x, y, z)$  is conservative.



## **XV. References**

Larson Hostetler, Edwards, Calculus with analytic geometry, Houghton Mifflin Company New York 1988.

## **XVI. Main Author of the Module**

The author of this module on Basic Mathematics was born in 1953 and went through the full formal education in Kenya. In particular he went to University of Nairobi from 1974 where he obtained Bachelor of Science (B.Sc) degree in 1977. A Master of Science (M.Sc.) degree in Pure Mathematics in 1979 and a Doctor of Philosophy (P.HD) degree in 1983. He specialized in the branch of Analysis and has been teaching at the University of Nairobi since 1980 where he rose through the ranks up to an Associate Professor of Pure Mathematics.

He has over the years participated in workshop on development of study materials for Open and Distance learning program for both science and arts students in which he has written books on Real Analysis, Topology and Measure Theory.

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