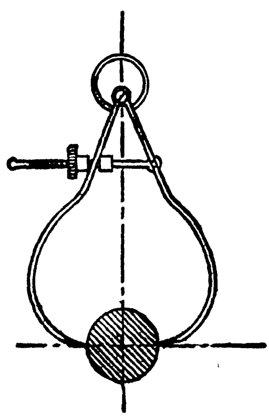
MATHEMATICS FOR MECHANICS

MATHEMATICS FOR MECHANICS

By

WILLIAM L. SCHAAF, Ph.D.

Assistant Professor of Education
Brooklyn College



GARDEN CITY PUBLISHING CO., INC.
GARDEN CITY, NEW YORK

MATHEMATICS FOR MECHANICS

By William L. Schaaf, Ph.D.
Assistant Professor of Education,
Brooklyn College

HERE AT LAST is a "math" book you can *really use* in your shop work. It will help you to solve hundreds of practical everyday problems—easily, quickly.

Mathematics for Mechanics is precisely what the title says. It gives you the knowledge you want—shop trigonometry, practical geometry, elements of algebra, and fundamental arithmetic. Simple, easy-to-understand instructions and hundreds of examples and illustrations show you plainly how to figure out any problem you need to, and *know* you are *right*.

If you know your "math" you are in line for bigger pay. That's the way to get singled out for promotion; it's mastery of mathematics that will make you able to supervise others.

The author of this book, Professor William L. Schaaf, of Brooklyn College, has had years of experience in teaching the kind of mathematics that mechanics need in their daily work. He knows just how to explain and illustrate every point and problem. As a result you can understand anything in

CONTENTS

CHAPTER

INTRODUCTION

WE ARE LIVING in an age in which mechanical arts and industrial production play a central role. The complexity of modern industry, as well as increased specialization in the various trades, have led to more exacting demands on the technical knowledge needed by the skilled mechanic. Prominent among these needs is an effective, working knowledge of mathematics. Indeed, it is no exaggeration to say that a mastery of basic mathematics is indispensable. The machine shop worker and the mechanic in almost every trade are constantly called upon to use certain fundamental mathematical facts and procedures. This is true not only of their daily activities on the job, but even more so when promotion and advancement are considered. The successful foreman is the man who, among other qualifications, can use his mathematics with confidence.

A number of practical, introductory books on mathematics have been written for laymen and for businessmen; but books on mathematics for *workmen* are less common. To be sure, many textbooks for machine shop students are available, but they generally include considerable specialized and advanced, difficult material. This book is precisely what the title says it is—a practical, introductory treatment of the mathematics essential for mechanics and workmen. In the shop and in the factory the worker must often perform tasks involving (1) computation, (2) measurement, (3) layout work, (4) interpretation of a blueprint, and (5) the use of tables and graphs. To carry out these activities, specific mathematical concepts and skills are required. A careful survey of the most important trade activities reveals which phases of arithmetic, algebra, geometry, and trigonometry will furnish the necessary mathematical facts and methods. It is these very phases which are encompassed here, constituting what might properly be called the "basic essentials" of the mathematics for workmen.

Experience shows that for the practical workman the most important parts of arithmetic include: the ability to use fractions and decimals with accuracy and facility; an understanding of ratio and proportion, and of percentage; familiarity with weights and measures; and a knowledge of the theory and use of precision measuring instruments.

As for the most useful skills of elementary algebra, these include the following: the use of symbols and formulas; positive and negative numbers; exponential notation; finding square roots; the solution of simple equations; the graphic representation of formulas and equations; the use of logarithms; and familiarity with the slide rule.

The contributions of geometry are also important. These include: the fundamental properties of common plane and solid geometric figures; the measurement of angles; the execution of certain basic geometric constructions; ability to measure and compute perimeters and areas of plane figures; ability to compute the surfaces and cubical contents and volumes of solid figures; the use of scale drawings; and an understanding of similar figures, both plane and solid.

Certain features of elementary plane trigonometry are most helpful, such as: the meaning of the sine, cosine and tangent of an angle; the use of trigonometric tables; the solution of right triangles; the trigonometric relations needed for the solution of oblique triangles; and methods of handling projections and special figures such as regular polygons.

In writing a book of this kind, two questions invariably arise: (1) how much shop information and trade knowledge should be included to illustrate the mathematical ideas and procedures, and (2) to what extent should the treatment be theoretical and explanatory, rather than merely a collection of empirical formulas and rule-of-thumb methods. Both of these problems, it is believed, have been happily solved. As for the first question, sufficient illustrative material has been drawn from the fields of physics, mechanics, drafting, machine shop, woodworking, metal trades, electrical trades and general engineering to demonstrate adequately the utility and actual application of the mathematical principles and processes, without, however, becoming a "handbook" or "textbook" in some particular trade. As for the second problem, *general methods* have been stressed in preference to specific methods or special tricks. Thus, for example, in the discussion of formulas and equations, skill in the transformation of formulas and the solution of equations in general has been emphasized. Or again, in the treatment of square root extraction, several methods and explanations are given, even though appropriate tables are furnished and recommended.

In conclusion, the author wishes to acknowledge his indebtedness to the following firms for permission to use published material: International Business Machine Corporation; Keufel and Esser Co.; McGraw-Hill Book Company; L. S. Starrett Company; and D. Van Nostrand Company.

W. L. S.

PART I

BASIC MATHEMATICS

CHAPTER I

FUNDAMENTALS OF ARITHMETIC

William L. Schaaf

---

1. WEIGHTS AND MEASURES

Importance of Measurement. In the mechanized civilization in which we live the vital importance of measurement cannot be overemphasized. Careful measurements are indispensable in business, in almost all trades, in manufacturing and industry, to say nothing of science, technology and engineering. A famous scientist, Lord Kelvin, once said that not until we can measure something can we really understand it. The same is true of design and production; finished products simply could not be made without the use of measurements. One has only to think of the many parts of an auto engine, the delicate mechanism of a watch, the sensitive shutter of a camera, or the precision of a lathe, to realize how utterly dependent modern man is upon the art of measurement. Indeed, the history of the development of units of measure reflects the march of civilization in more ways than one.

Nature of Measurements. To understand fully the nature and use of measurements, two basic principles must be borne in mind:

(1) *All measurements are approximate.*
(2) *All units of measure are arbitrary.*

There is no such thing as an absolute or perfect measurement. An object can be thought of as having an actual, real, or "true" length; but that length can never be found completely, it can only be found *approximately.* How "exact" any particular measurement happens to be depends upon the nature of the instruments used, the skill of the operator, and the conditions under which it is made. The difference between the true length and the measured length is technically known as the *error*. An error is not a mistake. The careless use, or the misuse, of a measuring instrument leads to mistakes. The proper use of a measuring instrument always involves errors. The errors may be large or small; they can never be completely eliminated. The extent of the approximation is known as the *degree of accuracy* of the measurement; a numerical measure of the extent of the error is known as the *precision* of the measurement. What particular degree of accuracy is sought depends chiefly upon the purpose for which the measurement is made, or the use to which the object is to be put. It would be a waste of time, for example, to measure the length of a fence post with the same care, i.e., with the same degree of accuracy, as used in measuring the length of a piston rod or a valve stem. Similarly, the accuracy used in measuring the length of a piece of molding in cabinet work would not suffice when measuring the diameter of needle valve. We shall learn more, however, about degrees of accuracy in Section 4 of this chapter.

Measurements are based upon certain *basic units* which are simply agreed upon by all concerned. In other words, the length of our present *inch* is what it is because it has been fixed by custom, or common consent; it has been *standardized,* so that it is always the same. But it might just as easily have been fixed, or standardized, at half its present length, or twice that amount, etc. In fact, at various times in history, and in different parts of the world, the inch has actually varied considerably from its present length. The same can be said of almost all the other units of measure that man has ever established or used: their values are *arbitrarily* fixed in accordance with conventional standards, and all who use these measures agree to abide by those standards.

Kinds of Measures. The fundamental measures are *length, mass,* and *time.* All other measures are derived from (or related to) one or another of these fundamental quantities. Thus *surfaces* (areas) and *volumes* (capacity) are measured in units derived from linear units; e.g., square inch and cubic inch. *Mass,* which is related to weight, is measured in the same units as weight, e.g., grams, ounces, pounds, etc. *Time* is measured in seconds, minutes, hours, etc. Other units of measure, such as those used for temperature, the electric current, or heat energy, are frequently more complicated. But whatever the unit used, it is always an arbitrary standard involving length, mass or time, or some combination of these. In this chapter we shall deal only with measures of length, area, capacity and weight.

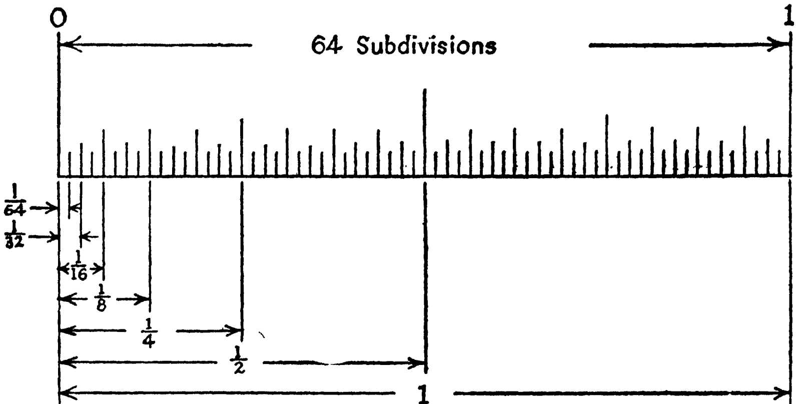
Standards of Measure. Today all basic units of measure used in this country, such as length, weight, area, capacity, temperature, etc., are determined by the United States Bureau of Standards in Washington, D.C. In the main, they agree with similar units used throughout the world. Two major systems are in use,—the *English* system and the *Metric* system, although the latter is little used in industry and trade, being the tool primarily of the scientist.

The best interests of all are served when the measuring instruments and devices used by scientists, engineers and industrial workers are sent to the Bureau of Standards at regular intervals to be checked for accuracy against universally accepted standards. Furthermore, most trades and industries have set up the standards which are used in their own respective fields. These standards concern the choice of measuring units, various specifications, and working conditions; they do not, of course, conflict with the Federal Government's basic standards. But cooperation among various industries, and in various fields of technology, long ago became so important, that the formation of the American Standards Association was inevitable. This organization passes on the standards set in various parts of the country and works with similar organizations in other countries to establish and maintain desirable standards of measures, specifications, and trade practices.

Linear Measure. In the English system, the commonly used linear measures are as follows:

	unit	=	1 inch
12	inches	=	1 foot
3	feet	=	1 yard
5½	yards	=	1 rod
320	rods	=	1 mile
1760	yards	=	1 mile
5280	feet	=	1 mile

The inch is sometimes subdivided by *halves,* sometimes by *tenths.* Each of these methods is here illustrated; both are commonly used by mechanics and industrial workers.



One Inch, magnified four times

When sub-divided by the decimal system, the subdivisions run as follows:

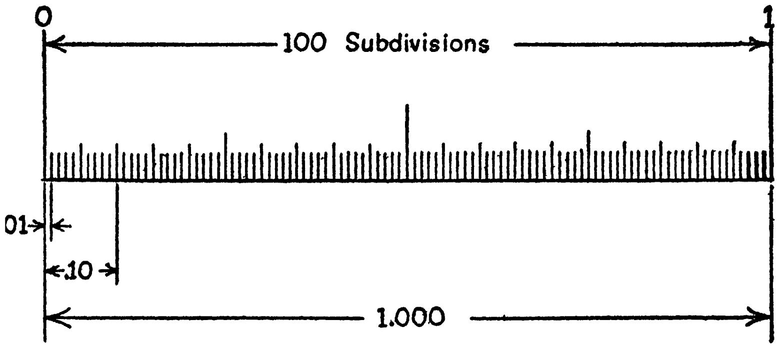
$$1 = 1.000 \quad \text{(one)}$$

$$\frac{1}{10} = .10 \quad \text{(one-tenth)}$$

$\frac{1}{100} = .01$ (one-hundredth)

$\frac{1}{1000} = .001$ (one-thousandth)

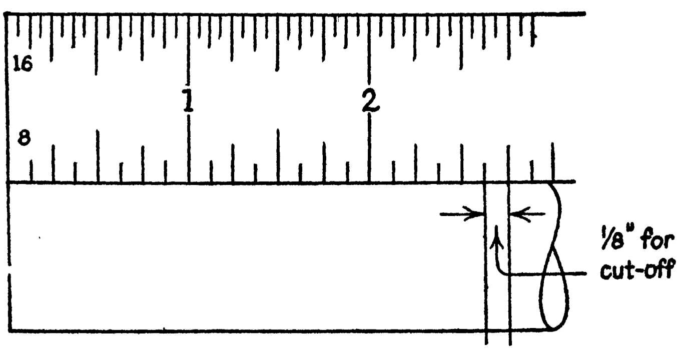
$\frac{1}{10,000} = .0001$ (one ten-thousandth)



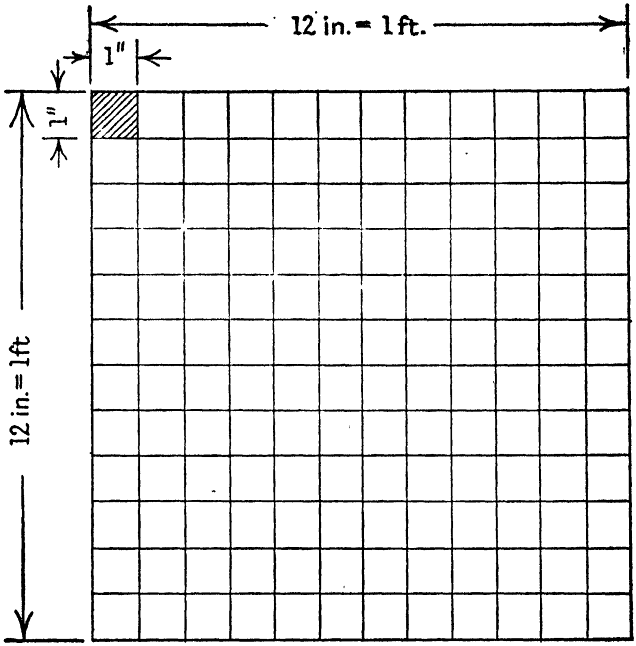
One Inch, magnified four times

Surface Measure. The commonly used measures of area in shop work and trade operations include:

unit	=1 square inch
144 square inches	=1 square foot
9 square feet	=1 square yard



Showing use of scale in measuring a 2½" piece to be cut from stock allowing 1/8" for cut-off.



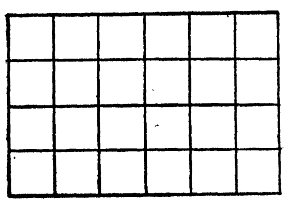
Pictorial representation, showing how
1 sq. ft. contains 144 sq. in.

A square inch is derived from the linear inch by taking as the unit of surface a square one inch on each side. The area of a surface is taken to be the number of square inches it contains. The area of a square is thus found by multiplying the length of one side by the other side, i.e.,

$$A = s \times s, \text{ or } A = s^2,$$

which is read "*area equals* s *squared.*" Similarly, the area of a rectangle is found by multiplying its length by its width; i.e.,

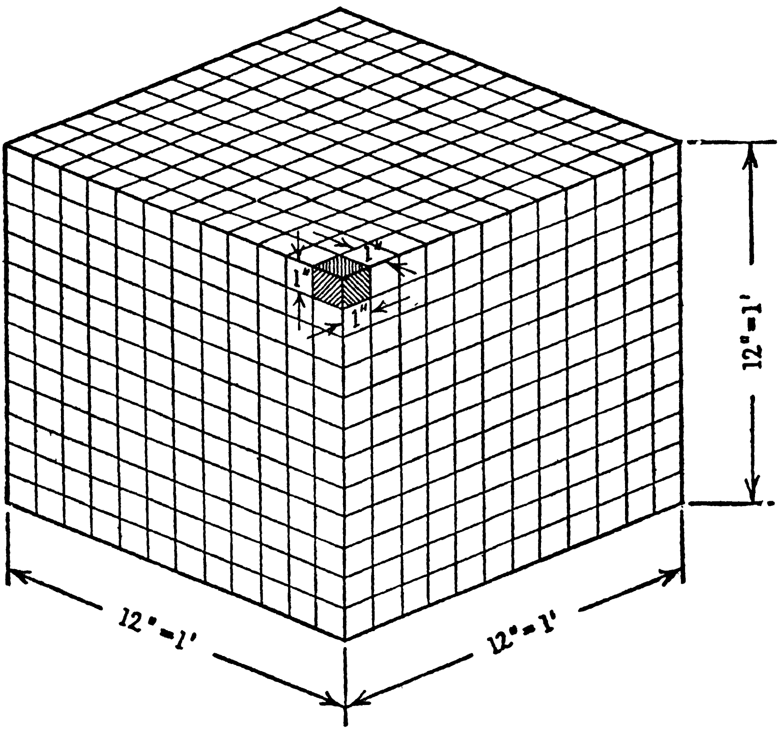
$$A = l \times w, \text{ or } A = lw.$$



Area 4 x 6 = 24 Sq. in.

Volume and Capacity. The *volume* of a solid object such as a block of wood or a slab of metal refers to the amount of space it occupies. The *capacity* of a hollow vessel, such as a glass jar, a wooden box, or a metal bucket, refers to the quantity of material it can contain or hold; capacity is therefore expressed in terms of "inside" measurements, since the thickness of the walls of the container cannot be disregarded (unless they are comparatively thin, or the measure of capacity is to be taken rather approximately). Volume and capacity are often measured in terms of the same units, viz.:

unit = 1 cubic inch
1728 cubic inches = 1 cubic foot
27 cubic feet = 1 cubic yard
1 cubic yard = 1 load



Pictorial representation, showing how
1 cu. ft. contains 1728 cu. in.

The unit of volume is taken as a cube each of whose edges is one inch in length. The volume of a cube is thus given by

$$V = e \times e \times e, \text{ or } V = e^3,$$

which is read *"volume equals e cubed."* Similarly, the volume of a rectangular solid of length l, width w and thickness (or height) h, is given by

$$V = l \times w \times h, \text{ or } V = lwh.$$

For certain purposes, capacity is also measured in other units as well. For liquid measure, we have:

4 gills	=	1 pint
2 pints	=	1 quart
4 quarts	=	1 gallon

Measures of Weight. For measuring the weight of common objects, several systems of measures are in common use, viz., the *Avoirdupois, Troy,* and *Apothecaries'* systems. For all ordinary purposes, and unless otherwise specified, weights are in the Avoirdupois system; Troy weight is used only by jewelers, etc., when weighing gold, silver and gems, while the Apothecaries' system is used only by pharmacists and physicians.

AVOIRDUPOIS WEIGHT

16 ounces (oz.)	=	1 pound (lb.)
100 pounds	=	1 hundredweight (cwt.)
20 hundredweights	=	1 ton (T.)
2000 pounds	=	1 ton
2240 pounds	=	1 long ton
7000 grains (gr.)	=	1 pound avoirdupois

Certain common "equivalents" are well worth remembering. These are:

1 gallon	=	231 cubic inches
1 cubic foot	=	7½ gallons
1 barrel	=	31½ gallons
1 cubic foot of water	=	62.4 pounds

Exercise 1.

1. A roll of paper is 88″ wide; what is the width expressed in feet?
2. Find the length of the border of a rectangular rug measuring 9′3″× 12′4″.
3. An airplane is traveling at a speed of 270 miles per hour; this is equivalent to how many feet per minute?
4. A piece of metal screening contains 20½ sq. yd.; how many square feet of surface will it cover?
5. A velocity of 88 ft. per second is equivalent to how many miles per hour?
6. A gallon of paint covers 150 sq. ft. with one coat. How many gallons will be required to cover a surface of 750 sq. ft. with two coats of this paint?
7. Linoleum costing 12¢ a sq. yd. was laid on a kitchen floor 12′×21′. What was the cost of the linoleum?
8. A point on the rim of a flywheel is moving at the rate of 1200 ft. per minute; what is the rate in feet per second?

9. A sheet of metal foil measures 2½"×4". How many sq. ft. of foil are there in 1000 such sheets?
10. How many cubic feet are there to a gallon? to the barrel?
11. How much does a gallon of water weigh?
12. If a roof tank contains 1500 gallons of water, what is the weight of its contents?
13. A fuel tank for an oil burner has a capacity of 300 gallons; how many cubic feet is this?
14. A vat in a soap factory has a capacity of 1600 cu. ft. If it is filled to ¾ of its capacity with liquor weighing 70 lb. per cu. ft., what is the weight of its contents?
15. An air blower in a factory has a rectangular cross section, 12"×20". Air is blown through the duct at the rate of 6 ft. per second. At this rate, how many cubic feet of air pass a given point every minute?

Metric System. This is the system of weights and measures used by scientists the world over. While it is legal and permissive in the United States, it has never been widely adopted in industrial and shop practice in this country.

The fundamental standard of length of the metric system is the meter, which is defined as the distance between two scratch-marks on a bar of platinum-iridium, carefully preserved at the International Bureau of Weights and Measures near Paris, when the temperature of the bar is that of melting ice (o° Centrigrade). All other units of length are multiples or submultiples of the meter, as shown below.

METRIC LINEAR MEASURE

10 millimeters (mm.)	=1 centimeter (cm.)
10 centimeters	=1 decimeter (dm.)
10 decimeters	=1 meter (m.)
10 meters	=1 decameter (Dm.)
10 decameters	=1 hectometer (Hm.)
10 hectometers	=1 kilometer (Km.)
10 kilometers	=1 myriameter (Mm.)

It should be added that the most commonly used of these units are the kilometer, the meter, the centimeter and the millimeter. For purpose of converting lengths and distances from one system to the other, the following approximate equivalents are convenient:

APPROXIMATE EQUIVALENTS

1 centimeter	=about 0.4 inch
1 meter	=about 1.1 yard
1 kilometer	=about 0.6 mile

1 inch	=about 2.5 centimeters	
1 yard	=about 0.9 meter	
1 mile	=about 1.6 kilometers	

Metric Surface Measure. The basic units of metric surface measure is the *square meter,* which is a square one meter long on a side. Another convenient and commonly used unit of area is the square centimeter, which is a square one centimeter on each side. The relations between the various units of area are given below.

METRIC SURFACE MEASURE

100 square millimeters (sq. mm.)	=1 square centimeter (sq. cm.)
100 square centimeters	=1 square decimeter (sq. dm.)
100 square decimeters	=1 square meter (sq. m.)
100 square meters	=1 square decameter (sq. Dm.)
100 square decameters	=1 square hectometer (sq. Hm.)
100 square hectometers	=1 square kilometer (sq. Km.)

For ordinary purposes and convenience in converting from English into metric units and vice versa, the following equivalents are given:

APPROXIMATE EQUIVALENTS

1 sq. inch	=about 6.5 sq. centimeters
1 sq. foot	=about .09 sq. meter
1 sq. yard	=about .84 sq. meter
1 sq. mile	=about 2.6 sq. kilometers
1 sq. centimeter	=about .15 sq. inch
1 sq. kilometer	=about .39 sq. mi.

Metric Measures of Capacity. In measuring cubical contents by the metric system the fundamental unit of volume is the *cubic meter.* Other units commonly employed are as follows:

METRIC UNITS OF VOLUME

1000 cubic millimeters (cu. mm.)	=1 cubic centimeter (cu. cm.)
1000 cubic centimeters	=1 cubic decimeter (cu. dm.)
1000 cubic decimeters	=1 cubic meter (cu. m.)

For measuring both liquids and solids, the *liter* is commonly employed A liter is the same as a cubic centimeter, or 1000 cu. cm. (also abbreviated *cc.*).

APPROXIMATE EQUIVALENTS

1 cubic inch	=about 16 cubic centimeters
1 cubic centimeter	=about .06 cubic inch
1 quart (liquid)	=about .95 liter
1 liter	=about 1.06 quart (liquid)

Metric Units of Weight. Compared to the variety of English units of weight, the metric system is doubtless much simpler. Here the standard unit of weight is the *kilogram,* which is defined as the weight of a certain mass of platinum-iridium kept at the International Bureau of Weights and Measures near Paris, and known as the International Prototype Kilogram. The kilogram and the gram are the two most widely used units, except for very large weights.

METRIC MEASURES OF WEIGHT

10 milligrams (mg.)	=1 centigram (cg.)
10 centigrams	=1 decigram (dg.)
10 decigrams	=1 gram (g.)
1000 grams	=1 kilogram (Kg.)
1000 kilograms	=1 metric ton (T.)

APPROXIMATE EQUIVALENTS

1 ounce=28.35 grams	1 gram=.035 ounce
1 pound=454 grams=.45 kilogram	1 kilogram=2.2 pounds

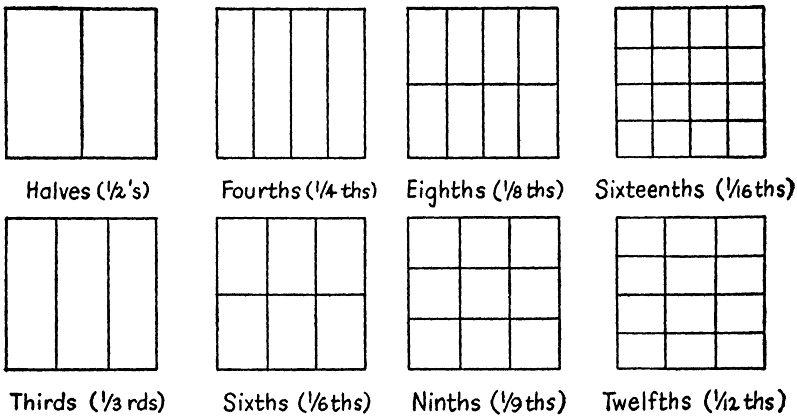
Exercise 2.

1. How many ounces are there in 250 grams?
2. How many inches are there in 45 millimeters?
3. How many centimeters are there in 15 inches?
4. How many kilograms are there in 12 pounds?
5. How many inches are there in 50 centimeters?
6. How many pounds are there in 75 kilograms?
7. How many millimeters are there in 6½ inches?
8. How many grams are there in 8 ounces?
9. What is the weight in kilograms of an athlete weighing 172 pounds?
10. What is the height in feet and inches of a man whose medical record states that he is 178 cm. tall?
11. How many cubic centimeters are there in a glass container capable of holding 2 quarts?
12. How many quarts can a 10-liter jug hold?
13. How many pints are there in a vessel whose capacity is 750 cc.?
14. How many liters are there in a 2-gallon chemical container?
15. How many sq. cm. are there in the surface of a metal plate having an area of 200 square inches?

2. COMMON FRACTIONS

Whole Numbers and Fractions. Numbers like 2, 5, 13, 40, etc., are known as whole numbers, or *integers.* When measurements are made, the magnitude is expressed, if possible, in some integral number of units, as:

3 inches, 5 feet, 12 pounds, etc. But with many magnitudes this is impossible, and it becomes necessary to make use of a *part of a unit.* To do this, the entire unit is *thought of* as being divided into any convenient number of *equal parts;* one or more of such equal parts into which a whole unit has been divided is known as a *fraction.* Thus $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{6}$ and $\frac{8}{10}$ are common fractions. The number below the fraction line, called the *denominator,* states into how many equal parts the whole unit has been divided; the number above the line, known as the *numerator,* tells how many of these equal parts are being considered, or measured, or used.



Fractions like $\frac{5}{8}$, $\frac{3}{16}$, $\frac{1}{4}$, and $\frac{15}{32}$, where the numerator is smaller than the denominator, are known as *proper fractions.* Proper fractions always indicate a quantity which is less than 1. On the other hand, fractions like $\frac{4}{3}$, $\frac{3}{2}$, $\frac{9}{8}$, $\frac{25}{12}$, where the numerator is greater than the denominator, are called *improper fractions;* such fractions always indicate an amount larger than 1, or one unit. Fractions like $\frac{2}{2}$, $\frac{3}{3}$, $\frac{8}{8}$, where the numerator equals the denominator, are also called improper fractions, although their value is exactly equal to 1 in each case. Strictly speaking, they do not represent a fractional part of a unit at all; they only have the form of a fraction, and actually represent the *whole unit,* since they say, in effect, "divide the whole unit into a certain number of equal parts, and then consider *all* of those parts."

An improper fraction of the first kind mentioned, e.g., $\frac{5}{3}$, can also be expressed as the sum of a whole number and a proper fraction; thus $\frac{5}{3}=\frac{3}{3}+\frac{2}{3}=1+\frac{2}{3}=1\frac{2}{3}$. When written in this final form, as an integer plus a proper fraction, but with the + sign omitted, it is usually called a *mixed number.*

EXAMPLE 1: Express $^{25}/_{8}$ as a mixed number.

SOLUTION: $^{25}/_{8}=\frac{24+1}{8}={}^{24}/_{8}+{}^{1}/_{8}=3+{}^{1}/_{8}=3{}^{1}/_{8}$, *Ans.*

EXAMPLE 2: Change $4^{7}/_{16}$ to an improper fraction.

SOLUTION: $4^{7}/_{16}=4+{}^{7}/_{16}=\frac{4\times16}{16}+\frac{7}{16}={}^{64}/_{16}+{}^{7}/_{16}={}^{71}/_{16}$, *Ans.*

Reduction of Fractions. Proper fractions like $^{5}/_{8}$, $^{9}/_{16}$, $^{3}/_{4}$, and $^{2}/_{3}$ are said to be in their *lowest terms.* This means that both the numerator and the denominator cannot be further reduced by dividing each of them by the same number. On the other hand, fractions like $^{6}/_{8}$, $^{10}/_{16}$, $^{8}/_{32}$, and $^{4}/_{64}$ are ***not*** in their lowest terms; they can be reduced further by dividing ***both*** numerator and denominator by some number which is an exact divisor of each of them; dividing both the numerator and denominator of a fraction by a common divisor is always permissible, since it does not alter the value of the original fraction. Thus:

$$^{6}/_{8}={}^{3}/_{4};\quad ^{10}/_{16}={}^{5}/_{8};\quad ^{8}/_{32}={}^{1}/_{4};\quad ^{4}/_{64}={}^{1}/_{16}.$$

Fractions can also be transformed by *multiplying* both numerator and denominator by any desired number, provided the *same* number is applied to both parts of the fraction; this does not change the value of the fraction either. It enables us, however, to express a fraction in any other denomination. Thus we have the following:

EXAMPLE 1: Change the fraction $^{7}/_{8}$ to an equivalent fraction having the denominator 32.

SOLUTION: Dividing 32, the new denominator, by 8, the old denominator, gives 4; this is the number by which we must multiply each part of the fraction.

$$^{7}/_{8}=\frac{7\times4}{8\times4}={}^{28}/_{32},\ \textit{Ans.}$$

EXAMPLE 2: Express $^{3}/_{4}$ as 64ths.

SOLUTION: $64\div4=16$

$$^{3}/_{4}=\frac{3\times16}{4\times16}={}^{48}/_{64},\ \textit{Ans.}$$

Exercise 3.

1. Change $^{3}/_{8}$ in. to 64ths; $2^{1}/_{4}$ in. to 16ths; $1^{5}/_{8}$ in. to 32nds; $10^{3}/_{16}$ in. to 64ths.

2. The thickness of a brass plate is $^{18}/_{32}''$; express this in eighths of an inch. Is its thickness more than $\frac{1}{2}''$? If so, how much more? Is it less than $\frac{3}{4}''$? If so, how much less?
3. Find the equivalent number of 16ths of an inch in $2\frac{1}{2}$ in.; in $1\frac{5}{8}$ in.; in $3\frac{3}{16}$ in.; in $\frac{3}{4}$ in.
4. A piece of cardboard is $5\frac{3}{4}$ in. wide. If it is to be cut into strips each $\frac{1}{4}$ in. in width, how many such strips will there be?
5. The face of a metal block is $3\frac{5}{8}$ in. wide; this is equivalent to how many 16ths? how many 32nds?
6. Is $^{18}/_{32}$ larger than $^{10}/_{12}$? Is $2\frac{1}{4}$ larger than $^{28}/_{16}$? Arrange the following dimensions in order of magnitude, beginning with the smallest: $\frac{7}{8}''$, $2\frac{1}{16}''$, $2\frac{3}{64}''$, $\frac{3}{4}''$.

Addition and Subtraction of Fractions. When adding two or more fractions having the same denominator, we simply add their numerators and place the result above their *common* denominator. Thus:

$$\frac{3}{8}+\frac{7}{8}+\frac{5}{8}=\frac{3+7+5}{8}=\frac{15}{8}=1\frac{7}{8}.$$

If the denominators of the fractions to be added are not all alike, the fractions must first be changed to equivalent fractions which do have the same denominator. The *least common denominator* (L.C.D.) is the *smallest denominator* that is exactly divisible by *each* of the denominators in question. For example:

(1) to add $\frac{3}{8}$, $\frac{3}{4}$ and $\frac{1}{2}$, the L.C.D.=8

(2) to add $\frac{2}{5}$, $\frac{3}{10}$ and $\frac{1}{4}$, the L.C.D.=20.

In practical problems, the L.C.D. is readily found by inspection.

Example 1: Add, $\frac{2}{3}+\frac{1}{2}+\frac{3}{4}$.

Solution: L.C.D.=12.

$$\frac{2}{3}=\frac{8}{12};\quad \frac{1}{2}=\frac{6}{12};\quad \frac{3}{4}=\frac{9}{12}$$

$$\frac{8}{12}+\frac{6}{12}+\frac{9}{12}=\frac{8+6+9}{12}=\frac{23}{12}=1\frac{11}{12},\ \textit{Ans.}$$

Example 2: Add, $2\frac{1}{4}+\frac{3}{16}+4\frac{7}{8}+\frac{5}{32}$

Solution: L.C.D.=32.

$$\frac{9}{4}+\frac{3}{16}+\frac{39}{8}+\frac{5}{32}$$

$$=\frac{72}{32}+\frac{6}{32}+\frac{156}{32}+\frac{5}{32}$$

$$=\frac{72+6+156+5}{32}=\frac{239}{32}=7\frac{15}{32},\ \textit{Ans.}$$

Example 3: Subtract $4\frac{3}{8}$ from $6\frac{5}{16}$.

Solution: L.C.D. = 16.

$$6\tfrac{5}{16} = 6\tfrac{5}{16} \qquad 5\tfrac{21}{16}$$

$$4\tfrac{3}{8} = 4\tfrac{6}{16} \qquad \underline{4\tfrac{6}{16}}$$

$1\frac{15}{16}$, *Ans.*

Exercise 4.

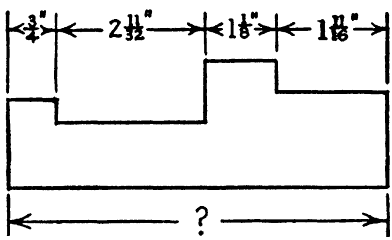
Add the following:

1. $\frac{2}{3}+\frac{3}{4}$
2. $\frac{1}{2}+\frac{5}{8}+\frac{1}{4}$
3. $\frac{1}{8}+\frac{7}{8}+\frac{3}{4}$
4. $\frac{3}{8}+\frac{1}{4}+\frac{5}{16}$
5. $2\frac{1}{2}+5\frac{7}{8}$
6. $3\frac{1}{4}+2\frac{1}{16}+1\frac{5}{8}$
7. $\frac{7}{8}+5\frac{1}{4}+2\frac{3}{16}$
8. $4\frac{1}{2}+\frac{5}{32}+2\frac{1}{8}+1\frac{3}{4}$

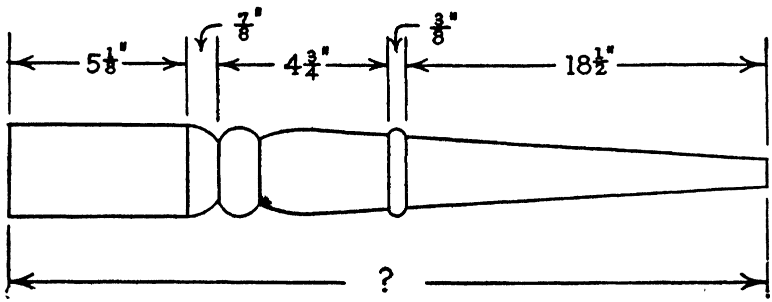
Subtract:

9. $3\frac{1}{2}$ from $5\frac{7}{8}$
10. $\frac{9}{16}$ from $2\frac{3}{4}$
11. $\frac{11}{64}$ from $\frac{9}{32}$
12. $3\frac{5}{8}$ from $5\frac{3}{32}$

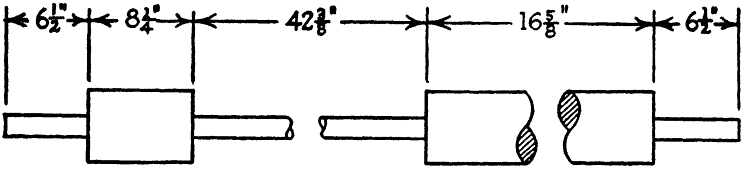
13. Find the overall length of the metal plate shown below:



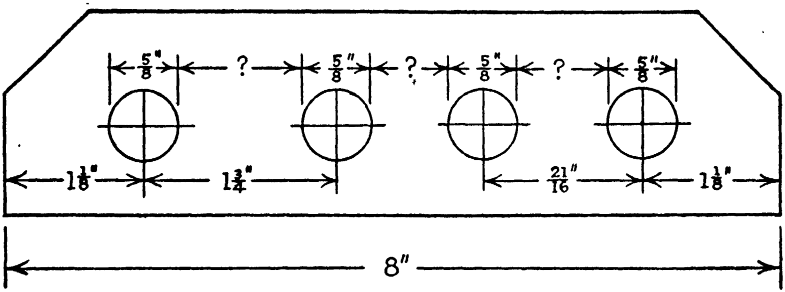
14. Find the total length of the wooden piece to be turned as follows:



15. What is the entire length of the shaft here shown?



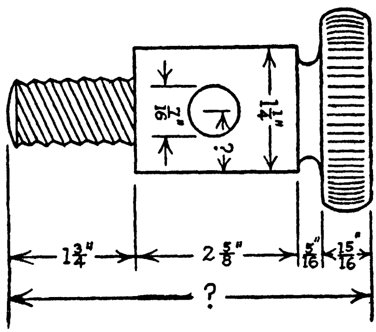
16. A metal plate is 1³⁄₁₆″ thick. If ¹⁄₆₄″ is removed from the top surface and ³⁄₃₂″ from the bottom surface, what is its final thickness?

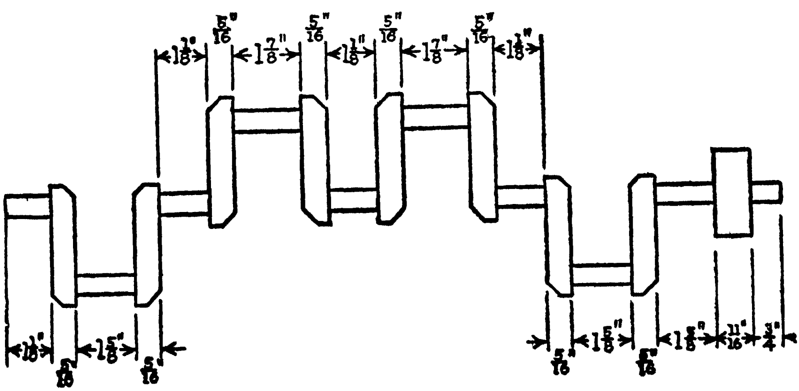


17. Find the missing dimensions in the wooden fixture with four holes drilled as per specification.

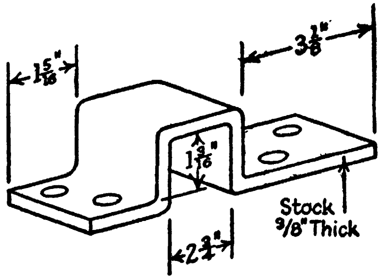
18. Find the overall length of the metal pin here shown; also, the distance from the center of the hole to the edge.

19. Find the overall length of the engine crank shaft here shown.





20. What length of stock is required to make the bent metal fixture here shown? (To allow for the bends, add ½ of the thickness of the stock for each right angle bend to the total of the inside measurements.)



Multiplication and Division of Fractions. In order to multiply two or more fractions together, the respective numerators are multiplied to find the numerator of the product, and their respective denominators are multiplied to find the denominator of the product; the resulting fraction is then reduced to lowest terms, if possible. Should some of the numerators have common factors, these should be divided out first (commonly called "cancellation";) the remaining factors are then multiplied together as before. If mixed numbers are to be multiplied, they are first changed to improper fractions.

EXAMPLE 1: Multiply $\frac{3}{4}\times\frac{5}{8}\times\frac{1}{2}$.

SOLUTION: $\frac{3}{4}\times\frac{5}{8}\times\frac{1}{2}=\frac{3\times5\times1}{4\times8\times2}=\frac{15}{64}$, *Ans.*

EXAMPLE 2: Multiply $\frac{9}{16}\times\frac{2}{3}\times\frac{4}{5}$.

SOLUTION: $\frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{16}}}\times\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}}\times\frac{\overset{1}{\cancel{4}}}{5}=\frac{3\times1\times1}{2\times1\times5}=\frac{3}{10}$, *Ans.*

EXAMPLE 3: Multiply $3\frac{1}{7}\times10\frac{1}{2}$.

SOLUTION: $\frac{\overset{11}{\cancel{22}}}{\cancel{7}}\times\frac{\overset{3}{\cancel{21}}}{\cancel{2}}=33$, *Ans.*

To divide any quantity (a whole number, a fraction, or a mixed number) by a fraction, the divisor is inverted and then multiplied by the dividend; the same holds true even if the divisor is a whole number of an improper fraction. To divide by a mixed number, first change it to an improper fraction, and then proceed as before.

EXAMPLE 1: Divide $\frac{4}{9}$ by $\frac{5}{6}$.

SOLUTION: $\frac{4}{9}\div\frac{5}{6}=\frac{4}{\underset{3}{\cancel{9}}}\times\frac{\overset{2}{\cancel{6}}}{5}=\frac{8}{15}$, *Ans.*

EXAMPLE 2: Divide $\frac{9}{16}$ by 3.

SOLUTION: $\frac{9}{16} \div 3 = \frac{9}{16} \div \frac{3}{1} = \frac{\overset{3}{\not 9}}{16} \times \frac{1}{\not 3} = \frac{3}{16}$, *Ans.*

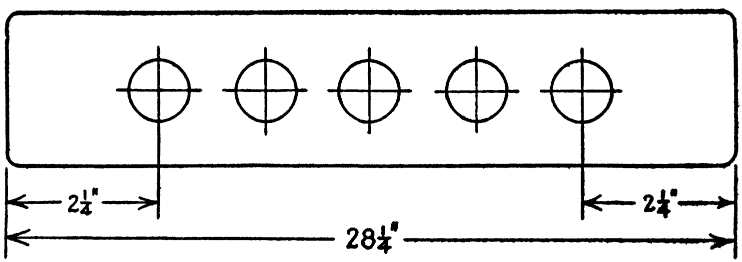
EXAMPLE 3. Divide $4\frac{2}{3}$ by $2\frac{1}{2}$.

SOLUTION: $4\frac{2}{3} \div 2\frac{1}{2} = \frac{14}{3} \div \frac{5}{2} = \frac{14}{3} \times \frac{2}{5} = \frac{28}{15} = 1\frac{13}{15}$, *Ans.*

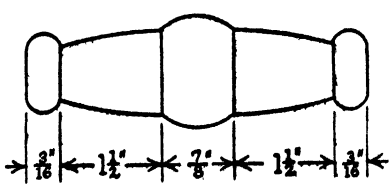
Exercise 5.

Perform the following indicated multiplications or divisions:

1. $1\frac{3}{8} \times 2\frac{1}{2}$
2. $24 \times 3\frac{5}{8}$
3. $3\frac{1}{4} \times 1\frac{1}{2} \times 16$
4. $12\frac{3}{8} \div 2\frac{1}{4}$
5. $14 \div 1\frac{3}{4}$
6. $3\frac{3}{4} \div 6$
7. If there are $7\frac{1}{2}$ gal. in a cubic foot, how many gallons are there in a tank whose capacity is $28\frac{1}{2}$ cu. ft. when it is $\frac{3}{4}$ filled?
8. If the five holes bored in the wooden strip here shown are to have their centers equally spaced, find the center to center distance between holes.



9. Eight pins each $4\frac{3}{8}''$ in length are cut from a piece of stock $38\frac{1}{4}''$ long. Allowing $\frac{1}{16}''$ waste for each cut, what is the length of the piece left?
10. In turning wooden handles like this one shown, $\frac{3}{4}''$ is allowed for waste on each handle. How many such handles may be made from a piece of stock, if the stock comes in 8 ft. lengths?



11. What is the weight of six $14\frac{1}{2}$-ft. lengths of pipe, if this particular size pipe weighs $3\frac{1}{4}$ lb. per running foot?
12. How many circular discs each $\frac{5}{16}''$ thick can be cut from a metal rod $36\frac{1}{4}''$ long, if $\frac{3}{32}''$ waste is allowed for each cut?

3. DECIMAL FRACTIONS

Meaning and Use of Decimal Fractions. Fractions with a denominator of 10, 100, 1000, etc., may be written as *decimals,* i.e., without a fraction line and without expressing the denominator in numbers; thus .3 is the same as $\frac{3}{10}$, 0.16 is the same as $\frac{16}{100}$, and .247 is the same as $\frac{247}{1000}$. In all decimal fractions the denominators are multiples of 10; they do not have to be written out, since the position of the decimal point takes the place of the denominator. Ordinarily, these decimals would be read as follows:

.3 as "three tenths."
.16 as "sixteen hundredths."
.247 as "two hundred forty-seven thousandths."
.3859 as "three thousand eight hundred fifty-nine ten-thousandths."

In many trades, measurements are sufficiently accurate when expressed in common fractions of an inch. However, not all fractional measurements in shop work are expressed as common fractions; as a matter of fact, in the machine shop, especially in precision work, the machinist uses decimal fractions of an inch more often than common fractions. The machinist, who uses precision instruments to attain the fine measurements called for, has developed a method of reading and designating decimals which differs somewhat from the expressions used by non-technical people. Studying the table given below will show you how to "talk decimals" in the machine-shop manner:

Decimal	*Designation.*
0.0001	One-tenth thousandth.
0.00025	One-quarter thousandth.
0.0005	One-half thousandth.
0.00075	Three-quarter thousandth.
0.001	One thousandth.
0.00125	One and one-quarter thousandths.
0.0015	One and one-half thousandths.
0.002	Two thousandths.
0.0025	Two and one-half thousandths.
0.003	Three thousandths.
0.0075	Seven and one-half thousandths.
0.010	Ten thousandths.
0.0125	Twelve and one-half thousandths.
0.015	Fifteen thousandths.
0.0156	Fifteen and six-tenth thousandths.
0.0312	Thirty-one and two-tenth thousandths.
0.1718	One hundred seventy-one and eight-tenth thousandths.

Exercise 6.

Write each of the following in figures:

1. Two hundred and seventy-six thousandths.
2. Fifteen and four-tenth thousandths.
3. Seven-tenth thousandths.
4. Four and one-quarter thousandths.
5. One hundred thousandths.

Write each of the following in words:

6. 0.3792	9. 0.0705	12. 0.4444
7. 0.0006	10. 0.2816	13. 0.0158
8. 0.2002	11. 0.0960	14. 0.00025

Addition and Subtraction of Decimals. These operations are carried out exactly as in the case of whole numbers, an important feature being the decimal points, which must be kept one under the other, including the decimal point in the sum or difference.

EXAMPLE 1: Add: 0.316+0.0592+1.8034+.26

SOLUTION:
$$\begin{array}{r} 0.316 \\ 0.0592 \\ 1.8034 \\ 0.26 \\ \hline 2.4386 \end{array}$$ *Ans.*

EXAMPLE 2: From 124.307, subtract 88.092.

SOLUTION:
$$\begin{array}{r} 124.307 \\ 88.092 \\ \hline 36.215 \end{array}$$ *Ans.*

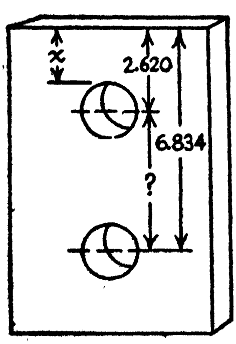
Another point that should be remembered with regard to adding or subtracting *measurements* involving decimals: never add or subtract measurements having different numbers of decimal places. Always "round off" the measurements, as required, to the same number of decimal places as appear in the measurement having the least number of decimal places, as shown below:

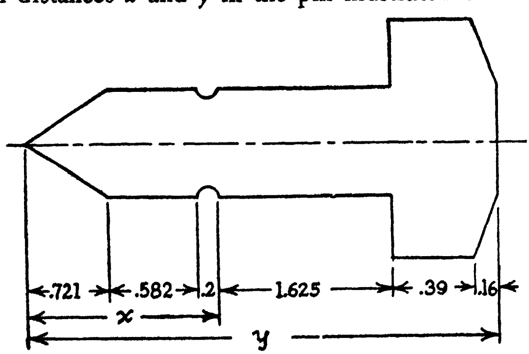
Incorrect		*Correct*	
8.36	inches	8.4	inches
10.082	"	10.1	"
4.5928	"	4.6	"
7.8	"	7.8	"
.749	"	.7	"
31.5838	inches	31.6	inches

If, however, we wish to add a measurement of 14¾″ to another measurement of 23½″, it is correct to say 14.75″+23.5″=38.25″, *provided that the second measurement is also taken accurately to two decimal places,* i.e., correct to the nearest .01 inch, in which case it should have been more properly expressed as 23.50″, showing that both the tenth's and hundredth's place had *actually been measured.* An exception to this statement may be made wherever dimensions on a drawing, or actual measurements on a piece of work, are understood to be taken to the same degree of accuracy; also when using precision gage blocks, as explained later in Section 4 of this chapter.

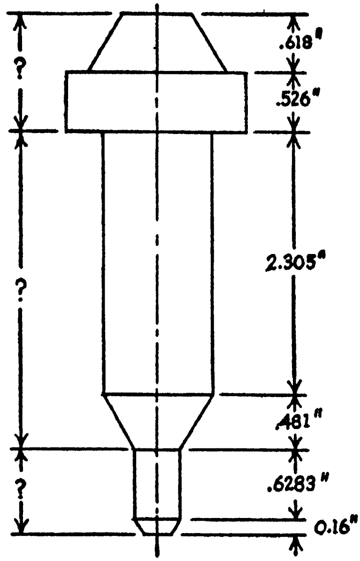
Exercise 7.

1. A hollow metal cylinder has a measured inside diameter of 2.0275″. If the wall of the cylinder is .245″ thick, what is the outside diameter?
2. A round piece of work is supposed to have a diameter of 2.375″. If it was turned .0008″ too large, what was the diameter?
3. Because of a mistake in dimensions on a blueprint a piece of work measuring 1.428 in. in thickness must be reduced 0.236 in. What will the thickness be after the reduction has been made?
4. It is desired to grind down a flat steel plate measuring 0.625″ thick by taking off .0075″. What will it measure after it has been ground?
5. A machinist milled off 0.184″ from each face of a circular brass plate. Before he had done this the plate measured 1.062″ in thickness. What did it measure afterward?
6. A toolmaker in checking a precision measurement uses four gage blocks measuring, respectively, as follows: .141″, .250″, 1.0007″ and 3.000″. Find the combined thickness of the four blocks.
7. Find the inside diameter of a circular tube whose outside diameter is 1.804″ and whose thickness is .216″.
8. Find the center distance between the two holes in the plate shown below. If the diameter of each hole is 1.62″, find the distance x.
9. Find the two overall distances x and y in the pin illustrated below.





10. The actual diameter of a crank shaft is 4.2836″. The inside diameter of the bearing into which this crank shaft fits is found to be 4.285″. What clearance does the shaft have?
11. The inside diameter of a hollow shaft measures 3.026″, and the outside diameter, 3.504″. Find the thickness of the shaft.
12. Find the indicated missing dimensions in the piece shown at the right.



Multiplication of Decimals. When a whole number is to be multiplied by a decimal, or when two decimals are to be multiplied together, the multiplication is first carried out exactly as with whole numbers; then, beginning at the right of the product, point off as many decimal places as there are in both factors together, prefixing ciphers if necessary.

EXAMPLE 1: Multiply 3.1416 by 32.

SOLUTION:

```
  3.1416
      32
  ------
   62832
  94248
--------
100.5312, Ans.
```

EXAMPLE: 2: Multiply .592 by .013.

SOLUTION:

```
   .592
   .013
   ----
   1776
   592
-------
.007696, Ans.
```

Exercise 8.

1. The length of the side of any square is always equal to the length of its side multiplied by 1.414. How long is the diagonal of a square 4.21 inches on a side?

2. The specific gravity of cast iron is 7.13, which means that cast iron is 7.13 times as heavy as an equivalent volume of water. If water weighs 62.425 lb. per cu. ft., what is the weight of 1 cu. ft. of cast iron?
3. A sheet of newspaper is .0031″ in thickness. What is the approximate thickness of a Sunday newspaper consisting of 220 pages?
4. The horizontal component of a force acting at an angle of 30° to the horizontal is equal to the force multiplied by .866. If the force amounts to 250 lb., what is the amount of the horizontal component?
5. A laminated piece is built up of 45 pieces of metal, each piece having a thickness of .0024″. How thick is the entire piece?
6. Gasoline is .91 times as heavy as water. If a quart of water weighs 2.08 lb., what is the weight of a gallon of gasoline?
7. A certain broach has 36 teeth. If each tooth cuts .018″, how much material is removed by the entire broach?
8. The specific heat of aluminum equals 0.218; this represents the number of calories required to raise 1 gm. of aluminum 1 degree Centigrade. How many calories are required to heat an aluminum block weighing 8.04 gm. from 20.4° C to 100.1° C?
9. A certain size steel bar weighs 7.656 lb. per linear foot. Find the cost of 2000 ft. of this bar, if the price is $2.20 per 100 lb.
10. The coefficient of expansion of iron equals .00000672 per degree Fahrenheit, which means that for each degree rise in temperature it increases in length by that fractional part of its original length. By how much will the length of a 200 ft. iron cable increase if the temperature rises 80° F.?

Division of Decimals. In order to divide a number by a decimal, the decimal point in both the divisor and the dividend must be moved as many places to the right as there are decimal places in the divisor.

EXAMPLE: Divide 92.862 by 2.91.

SOLUTION:

```
         31.91
    291)9286.20        Ans., 31.91+
        873
        ---
         556
         291
         ----
         2652
         2619
         ----
           330
           291
           ---
```

Exercise 9.

1. If one cubic foot contains 7.48 gal., find, to the nearest tenth, the number of cubic feet occupied by 550 gallons.

2. A certain size metal rod weighs 2.84 lb. per linear foot. How many feet of these rods are there in a bundle of various lengths, if the entire bundle weighs 86.75 lb.?
3. There are .3937 inches to one centimeter. How many centimeters long is a wire measuring 6.54″? (Carry the result to two decimal places.)
4. How many metal discs 0.0625″ thick can be stacked to a height of $2\frac{1}{4}$″?
5. A steel rod 38.26″ long is to be cut into 7 equal parts. Allowing 0.032″ for the thickness of each cut, how long will each piece be? (Remember that only six cuts need be made.)
6. In order to find the number of screws in a box, a mechanic weighs the entire box full of screws and finds the weight to be $2\frac{1}{4}$ lb. He also finds that a dozen screws weigh .28 lb. Making no allowance for the weight of the box, how many screws does it contain?

Changing Common Fractions to Decimals. Any common fraction may readily be changed to an equivalent decimal fraction simply by annexing zeros to the numerator and dividing by the denominator, as shown below; if the division does not terminate, it may be carried to as many decimal places as desired.

Example 1: Change $\frac{7}{23}$ to a decimal.

Solution:

```
     .30434
  23)7.00000        Ans., .30434+
     6 9
       100
        92
         80
         69
         110
          92
```

Example 2: Reduce $\frac{9}{32}$ to an equivalent decimal.

Solution:

```
     .28125, Ans.
  32)9.0000
     6 4
     2 60
     2 56
        40
        32
         80
         64
         160
         160
```

Changing Decimals to Common Fractions. To change a given decimal to a common fraction it is merely necessary to rewrite it with the denominator expressed as 10, 100, 1000, 10,000, etc., and reduce it to lowest terms if possible.

EXAMPLE 1: Change .042 to a common fraction.

SOLUTION: $.042 = \frac{42}{1000} = \frac{21}{500}$, *Ans.*

EXAMPLE 2: Reduce .6784 to a common fraction.

SOLUTION: $.6784 = \frac{6784}{10,000} = \frac{1696}{2500} = \frac{424}{625}$, *Ans.*

Exercise 10.

Change each of the following to decimal fractions:

1. $\frac{1}{8}$	5. $\frac{9}{32}$	9. $3\frac{1}{7}$	13. $\frac{11}{64}$
2. $\frac{3}{32}$	6 $\frac{15}{16}$	10. $\frac{5}{9}$	14. $\frac{49}{64}$
3. $\frac{7}{16}$	7. $\frac{27}{32}$	11. $\frac{12}{13}$	15. $\frac{5}{24}$
4 $\frac{15}{64}$	8. $\frac{11}{16}$	12. $\frac{2}{11}$	16. $\frac{25}{128}$

Change each of the following to common fractions:

17. .4375	20. .003125	23. .078125
18. .8125	21. .40625	24. .921875
19. .0625	22. .90625	25. .531250

Table of Decimal Equivalents. Since measurements on blueprints and in the shop are commonly expressed both as decimals and as ordinary fractions, it is important to be able to convert from one to the other quickly and easily. It is therefore highly desirable that some of the decimal equivalents should be memorized, especially the following:

$\frac{1}{2} = 0.500$	$\frac{1}{16} = 0.0625$
$\frac{1}{4} = 0.250$	$\frac{1}{32} = 0.03125$
$\frac{1}{8} = 0.125$	$\frac{1}{64} = 0.0156$

For further convenience the Table of Decimal Equivalents given herewith is constantly used by draftsmen and machinists.

Tolerance. Architects, carpenters and patternmakers generally use eighth's, sixteenth's, and thirty-second's instead of decimals. But they use them to different degrees of accuracy; thus the patternmaker rarely uses fractions of less than $\frac{1}{16}''$; the carpenter or cabinetmaker generally does not use fractions of less than $\frac{1}{8}''$; and so on. However, when it comes to making dies, tools and machine parts, great accuracy is required; for this purpose decimal fractions are more convenient and more useful. The term *tolerance* is used to indicate the limits within which a piece of work is acceptable when it deviates from the dimension indicated.

TABLE OF DECIMAL EQUIVALENTS

1/64	.015625	33/64	.515625
1/32	.03125	17/32	.53125
3/64	.046875	35/64	.546875
1/16	.0625	9/16	.5625
5/64	.078125	37/64	.578125
3/32	.09375	19/32	.59375
7/64	.109375	39/64	.609375
1/8	.1250	5/8	.6250
9/64	.140625	41/64	.640625
5/32	.15625	21/32	.65625
11/64	.171875	43/64	.671875
3/16	.1875	11/16	.6875
13/64	.203125	45/64	.703125
7/32	.21875	23/32	.71875
15/64	.234375	47/64	.734375
1/4	.2500	3/4	.7500
17/64	.265625	49/64	.765625
9/32	.28125	25/32	.78125
19/64	.296875	51/64	.796875
5/16	.3125	13/16	.8125
21/64	.328125	53/64	.828125
11/32	.34375	27/32	.84375
23/64	.359375	55/64	.859375
3/8	.3750	7/8	.8750
25/64	.390625	57/64	.890625
13/32	.40625	29/32	.90625
27/64	.421875	59/64	.921875
7/16	.4375	15/16	.9375
29/64	.453125	61/64	.953125
15/32	.46875	31/32	.96875
31/64	.484375	63/64	.984375
1/2	.5000	1.	1.0000

EXAMPLE 1: If the diameter of a round piece as indicated is 2.138″, with a tolerance of ±.003″; what are the limits within which it is acceptable?

SOLUTION:
$$\left.\begin{array}{l}2.138+.003=2.141\\2.138-.003=2.135\end{array}\right\}Ans.$$

Thus any dimension less than 2.141″ and more than 2.135″ is acceptable. Such a dimension might be written as 2.138±.003.

Example 2: What are the limits of measurement on a piece that calls for $3\frac{5}{16}''$, plus or minus $.010''$?

Solution: $3\frac{5}{16} = 3.3125$

$$\left.\begin{array}{l} 3.3125 + .01 = 3.3135'' \\ 3.3125 - .01 = 3.3115'' \end{array}\right\} \textit{Ans.}$$

Example 3: Using the table of decimal equivalents, change a fractional measurement of $.714''$ to the nearest 32nd of an inch.

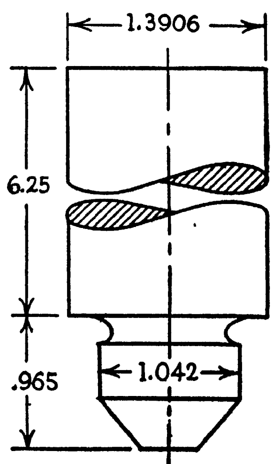
Solution: From the table,

$\frac{23}{32} = .71875$
$\frac{22}{32} = \frac{11}{16} = .6875$
$\frac{24}{32} = \frac{3}{4} = .75$

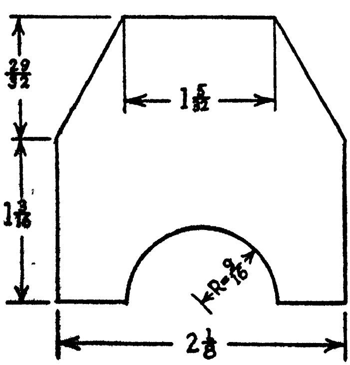
thus $.714 = \frac{23}{32}$, approx., *Ans.*

Exercise 11.

1. Using the table, change the following decimal dimensions to the nearest 32nd of an inch:
   a) .216 b) .8391 c) .10123
2. Change the following to the nearest 64th of an inch:
   a) .4869 b) .17235 c) .89268
3. In the round piece shown, find the allowable limits for each of the given dimensions if the tolerance is ±.004.
4. Find the limits for each dimension of the plate shown, if the tolerance allowed is ±.025.



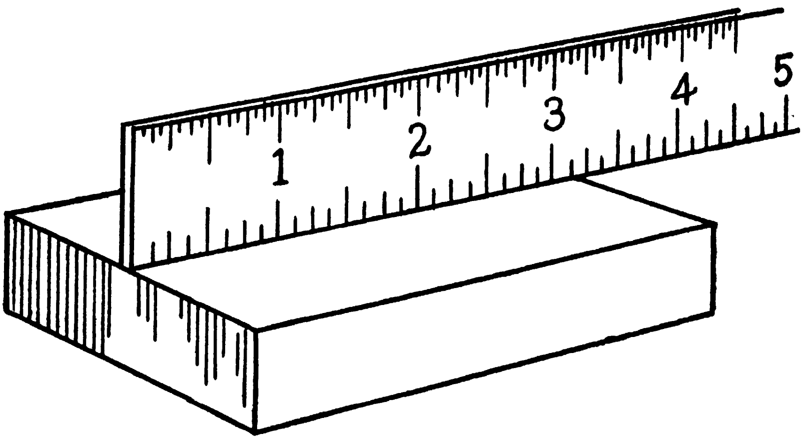
Ex. 3



Ex. 4

4. MEASURING INSTRUMENTS

Degree of Accuracy. As we saw in Section 1, all measurements are approximations, and the degree of accuracy of measurements may vary considerably. Extreme accuracy is not always required in shop work, even in machine shop operations. The greater the degree of accuracy achieved, the greater is the cost of the operation; it is uneconomical to secure greater accuracy than is actually needed. Where ultimate extreme accuracy is de-



sired, the degree of accuracy varies with the successive steps in the operation. Thus rough machining, finishing machining, grinding and lapping are often used in succession to achieve a final high degree of accuracy. For rough machining a steel scale would be used for making the measurement, which would be made to within $\pm\frac{1}{32}''$. For the second, or finishing machining, a micrometer might be used, reading the measurement to $\pm\frac{1}{64}''$. For the grinding operation a precision micrometer would be used, graduated in .0001″, and the measurement would be taken to within ±.0002″. During the lapping operation and for the finished part, gage blocks and indicators would be employed.

Limits of Accuracy. Scientists often use extremely precise measurements; the wave length of sodium light, for example, is 0.00005893 cm., which represents accuracy to 8 decimal places. For most industrial and shop operations, however, the limit of accuracy required is usually to the fourth decimal place. As we have already seen, the limits of accuracy required are referred to as the *tolerances,* and are generally specified on the blueprint or in the specifications which accompany the designs; they are just as important as the dimensions or measurements themselves. The American Standards Association has standardized tolerances for various parts, such as screw threads, cylindrical fits, and surface finish.

The limits of accuracy obtainable in making a measurement depend upon the nature of the particular instrument used, the conditions under which the measurement is made, and the skill of the operator. So far as the possible limitations of the measuring instrument are concerned, the following should be noted. The common steel scale has a limit of 1/64″ or 1/100″. The micrometer caliper will yield an accuracy of .001″, and, with a vernier attachment, to .0001″. These figures represent graduations on the scales of the instruments. The use of a toolmaker's miscroscope, or magnifying glass, is sometimes required when measuring to a graduated line, since the width of the line itself is approximately .006″. Ordinary precision gage blocks measure to .000008 of an inch, and the finest grade blocks to .000002 of an inch. Such blocks have no graduations, but are fixed in measurement; they are used in combination as will be explained below.

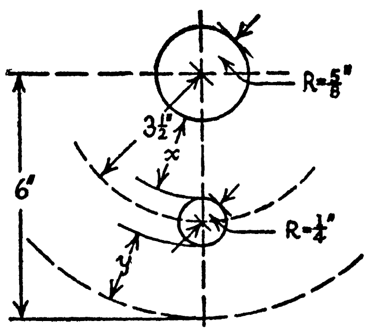
Conditions under Which Instruments Are Used. The degree of accuracy obtainable depends also upon the quality of the instrument and the working conditions. A particular instrument, even when new, will vary in manufactured accuracy, depending upon the quality and cost of the tool; an inexpensive, poorly made scale or calipers is never as accurate as a high grade, carefully made instrument. Again, as a tool becomes worn with constant use, it loses in accuracy. The lines marking the graduations on the scale become obliterated, the edge becomes nicked, and the moving parts (if any) become loose, and bearings are thrown out of alignment by knocking; all of these conditions diminish the accuracy obtainable. Finally, when working to four-place accuracy, temperature variations affect the accuracy of a measurement, since with metals particularly, both the object and the instrument are subject to expansion and contraction with a rise or fall in temperature. Thus for very accurate work the temperature should be ordinary room temperature, i.e., from about 68° to 72° Fahrenheit.

The Human Equation. Personal factors also influence accuracy. These include such considerations as normal eyesight and proper lighting; skill and care in estimating the smallest subdivision of a scale; correct habits, such as reading a meniscus properly, or avoiding parallax; and a delicate sense of touch, the ability to "feel" measure on a measuring instrument. Meticulous care in handling tools, their skilful use, and dependable judgment that comes only with experience—all these are required in making careful measurements. Lathes, shapers, milling machines, drills, taps, etc., are so designed and constructed that tolerances of ±.001″ or better can be achieved if skill and care are used. When extreme accuracy is required, unusual care must be exercised. Thus if a micrometer jaw is set too tightly, as much as .0005″ can be "forced"; or again, when reading a vernier scale, undue pressure against the sliding jaw may cause deflection of the object being measured, or slight clearances in the bearing surfaces of the sliding jaw, either of which will affect the reading of the instrument adversely.

Mechanical Duplication. The possibility of securing mechanical duplication depends not only upon the measuring instruments and the skill of the operator, but also upon the materials used and the machines and tools involved. Stock varies in quality and composition, in texture and finish, according to the particular shipment; such variations increase the difficulties in producing parts exactly alike. Similarly with equipment: the rigidity of a machine, the tightness of the bearings, the solidity of the machine bed, the fastening of fixtures, the wear and support of tools —all these are also involved in mechanical duplication.

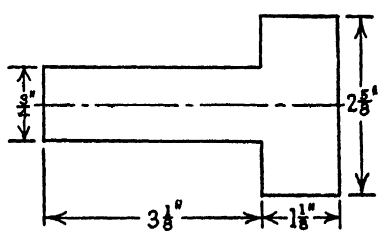
Exercise 12.

1. How much must be removed from a metal piece to be turned, if the original diameter is 3.764″ and the finished diameter required is 2.856″? if the original diameter is 1.827″ and the finished diameter is 1.792″?
2. Find the distances x and y, respectively, *before* the holes are reamed. If 1/64″ is allowed for reaming, what are these distances *after* reaming?



Ex. 2

3. The piece shown is to be rough turned, allowing 1/16″ for finish machining on all diameters and 1/32″ on all faces; find the corresponding dimensions for rough turning.



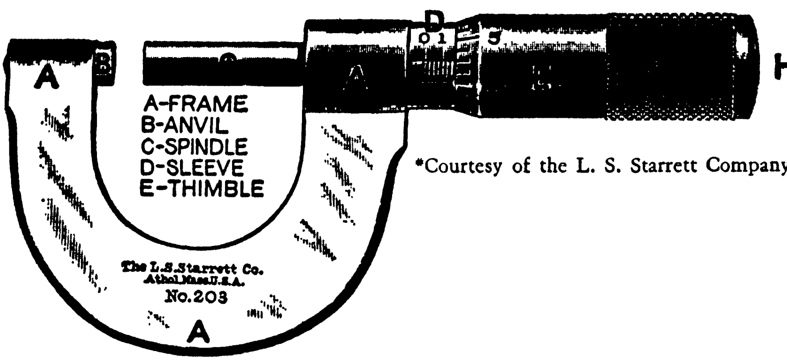
Ex. 3

4. The same piece (Ex. 3) is to be finish turned to allow .012″ for grinding on all diameters and .008″ on all faces; find the corresponding dimensions for finish machining.
5. If the required dimensions of a finished piece of work must be .462±.002, and the piece now measures .467, how much must still be removed?

The Micrometer. The *micrometer caliper,* or "mike," is the most common "precision instrument" used in the machine shop. Every machinist and toolmaker carries a micrometer. This instrument has many advantages:

1. It is small, and easily carried in the pocket.
2. It is convenient to handle and easy to read.
3. It is rugged enough to stand considerable handling.
4. It retains its accuracy, and has adjustments to compensate for wear.
5. It has a practical range of measurement, generally up to one inch.
6. It is comparatively inexpensive.

Construction and Use of the Micrometer.* The spindle C is attached to the thimble E, on the inside, at the point H. The part of the spindle which is concealed within the sleeve and thimble is threaded to fit a nut in the frame A. The frame being held stationary, the thimble E is revolved by the thumb and finger, and the spindle C, being attached to the thimble, revolves with it, and moves through the nut in the frame, approaching or receding from the anvil B. The article to be measured is placed between the anvil B and the spindle C. The measurement of the opening between the anvil and the spindle is shown by the lines and figures on the sleeve D and the thimble E.



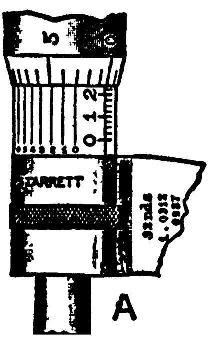
*Courtesy of the L. S. Starrett Company.

The pitch of the screw threads on the concealed part of the spindle is 40 to an inch. One complete revolution of the spindle therefore moves it longitudinally one-fortieth (or twenty-five thousandths) of an inch. The sleeve D is marked with 40 lines to the inch, corresponding to the number of threads on the spindle. When the caliper is closed, the beveled edge of the thimble coincides with the line marked 0 on the sleeve, and the 0 line on the thimble agrees with the horizontal line on the sleeve. Open the caliper by revolving the thimble one full revolution, or until the 0 line on the thimble again coincides with the horizonal line on the sleeve; the distance between the anvil B and the spindle C is then $\frac{1}{40}$ (or .025) of an inch, and the beveled edge of the thimble will coincide with the second vertical line on the sleeve. Each vertical line on the sleeve indicates a distance of 1/40 of an inch. Every fourth line is made longer than the others, and is numbered 0, 1, 2, 3, etc. Each numbered line indicates a distance of four times 1/40 of an inch, or one-tenth.

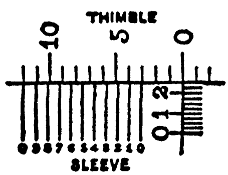
The beveled edge of the thimble is marked in twenty-five divisions, and every fifth line is numbered from 0 to 25. Rotating the thimble from one of these marks to the next moves the spindle longitudinally 1/25 of twenty-five thousandths or one-thousandth of an inch. Rotating it two divisions indicates two thousandths, etc. Twenty-five divisions will indicate a complete revolution, .025 or 1/40 of an inch.

To read the caliper, therefore, multiply the number of vertical divisions visible on the sleeve by 25, and add the number of divisions on the bevel of the thimble, from 0 to the line which coincides with the horizontal line on the sleeve. For example, as the tool is represented in the engraving, there are seven divisions visible on the sleeve. Multiply this number by 25, and add the number of divisions shown on the bevel of the thimble, 3. The micrometer is open one hundred and seventy-eight thousandths. ($7 \times 25 = 175 + 3 = 178$.)

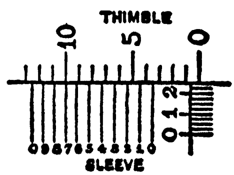
Using a Micrometer Graduated in Ten-Thousandths of an Inch. Readings in ten-thousandths of an inch are obtained by the use of a vernier, so named from Pierre Vernier, who invented the device in 1631. As applied to a caliper this consists of ten divisions on the adjustable sleeve, which occupy the same space as nine divisions on the thimble. The difference between the width of one of the ten spaces on the sleeve and one of the nine spaces on the thimble is therefore one-tenth of a space on the thimble. In engraving B the third line from 0 on thimble coincides with the first line on the sleeve. The next two lines on thimble and sleeve do not coincide by one-tenth of a space on thimble; the next two, marked 5 and 2, are two-tenths apart, and so on. In opening the tool, by turning the thimble to the left, each space on the thimble represents an opening of one-thousandth of an inch. If, therefore, the thimble be turned so that the lines marked 5 and 2 coincide, the caliper will be opened



A



B

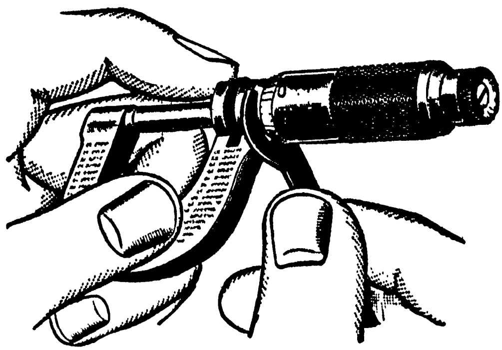


C

two-tenths of one-thousandth or two ten-thousandths. Turning the thimble further, until the line 10 coincides with the line 7 on the sleeve, as in engraving C, the caliper has been opened seven ten-thousandths, and the reading of the tool is .2507.

To read a ten-thousandths caliper, first note the thousandths as in the ordinary caliper, then observe the line on the sleeve which coincides with a line on the thimble. If it is the second line, marked 1, add one ten-thousandth; if the third, marked 2, add two ten-thousandths, etc.

Adjusting the Micrometer. These calipers will read correctly if there is no dirt between the anvil and spindle. When it becomes necessary to readjust the tool to compensate for the wear of screw and nut, this is done, not by the anvil, but by means of our friction sleeve, as follows: Take up the wear of screw and nut, then remove all dirt from face of the anvil and spindle and bring them together carefully. Insert the small spanner wrench in the small hole and turn until the line on the sleeve coincides with the zero line on the thimble.



Exercise 13.

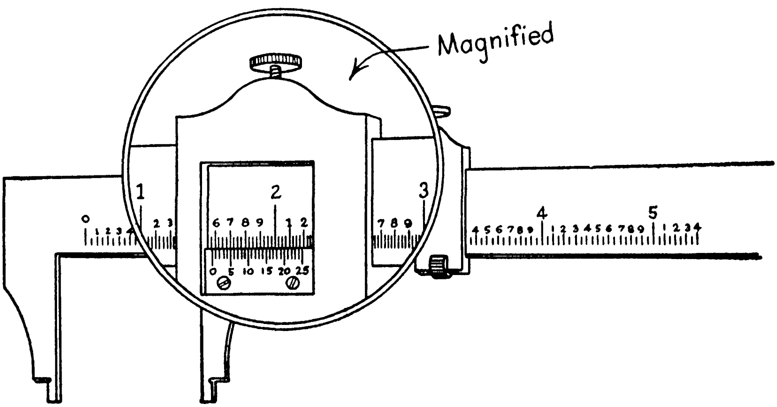
1. The following readings are taken on an ordinary micrometer; complete the table:

	Reading on sleeve is between	Nearest line on thimble	Complete Reading
(a)	.450 and .475	15	
(b)	.900 and .925	4	
(c)	.575 and .600	20	
(d)	.125 and .150	7	
(e)	.000 and .025	22	
(f)	.975 and 1.000	16	

2. The following readings are taken on a micrometer graduated in ten-thousandths of an inch; complete the table:

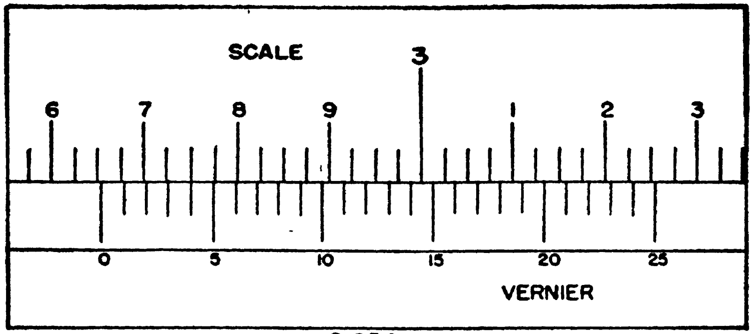
	Reading on sleeve is between	Reading on thimble between	Vernier line	Complete Reading
(a)	.325 and .350	22 and 23	3	
(b)	.875 and .900	8 and 9	9	
(c)	.450 and .475	16 and 17	2	
(d)	.200 and .225	11 and 12	4	
(e)	.575 and .600	2 and 3	6	
(f)	.600 and .625	24 and 25	1	

Vernier Instruments. The principle of the vernier has been applied to many kinds of instruments. In the foregoing discussion, it was seen how it was employed to increase the limit of measurement of the micrometer. Where the vernier is employed on an instrument as the sole agent for magnifying ordinarily imperceptible differences in length, it is usually known as a vernier caliper, vernier protractor, etc. It consists of a small auxiliary scale having usually one less or more graduations in the same length as the longer true scale. It is evident therefore that if the whole vernier scale contains one more division than the true scale over an equal length, each division on the vernier scale is proportionally

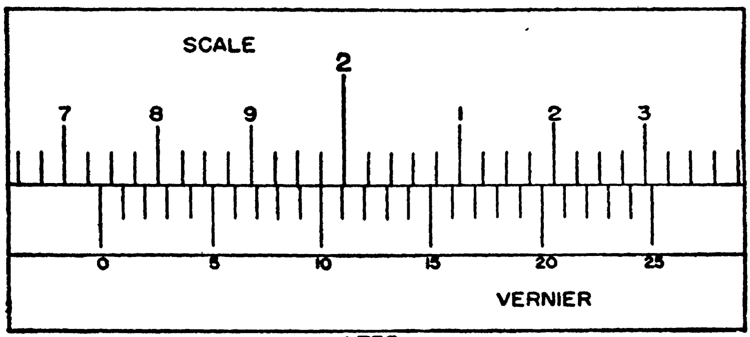


The Vernier Scale

smaller than a corresponding division on the true scale. If 25 divisions on the vernier scale are equal to 24 divisions on the main scale, then each division on the vernier scale is 1/25 of a division smaller than a division on the main scale. If there is an accumulating difference of 1/25 of a division, the effect of going along the vernier scale one division is to subtract 1/25 of a true scale division. By going along the scale 2, 3, 4, 5, etc., divisions of

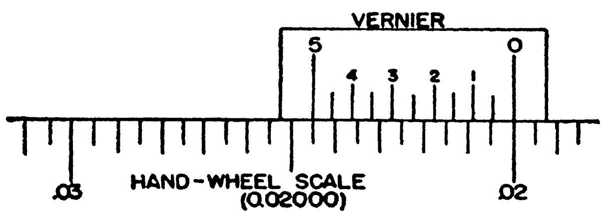


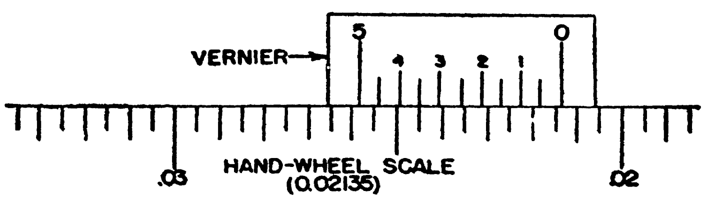
2.654



1.735

FROM A VERNIER CALIPER





FROM HAND-WHEEL OF JIG BORER

VERNIER SCALE READINGS

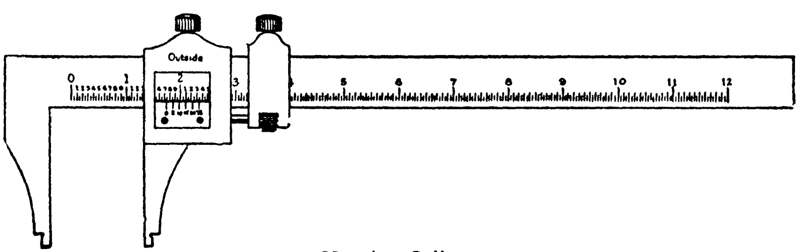
2/25, 3/25, 4/25, 5/25, etc., are subtracted from the original setting until the lines coincide. At this point all of the remaining fraction of a division indicated by the "0" on the vernier scale has been absorbed, and the number of the vernier divisions indicates the number of the 25ths this fraction of a division contains.

Vernier scales are not necessarily 25 units long; they may have any number of units. They may have only ten units, as on the vernier scale of the ten-thousandths micrometer. The graduated hand wheels of a machine tool such as a jig borer often employ the vernier scale for the purpose of indicating "tenths" or "half-tenths" of a thousandth of an inch table travel, etc.

Types of Vernier Instruments. The vernier scale has been applied to a variety of instruments and tools; for example:

1. Vernier Caliper
2. Vernier Height Gage
3. Vernier Depth Gage
4. Vernier Protractor

The typical vernier caliper consists of an L-shaped frame, the end of which is one of the jaws. On the long arm of the "L" is scribed the true scale, which may be 6, 12, 24, 36, or 48 inches long. The sliding jaw carries a vernier scale on either side. The scale on the front side is for outside measurements, whereas the scale on the back side is for inside measurements. It will be noted in the figure that the tips of the jaws have been formed so as to be capable of making an inside measurement. The thickness of the measuring points is automatically compensated for on the inside scale. The sliding jaw assembly consists of two sections joined by a horizontal screw. By clamping the right-hand section at its approximate location, a final fine adjustment of the movable jaw may be obtained by turning the adjusting nut. The sliding jaw may be clamped in any position with the locking screw shown in the figure on top of the jaw. The jaws of all vernier calipers, except the larger sizes, have two center points which are particularly useful in setting dividers to exact dimensions.

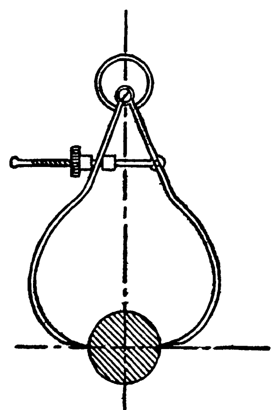


Vernier Caliper

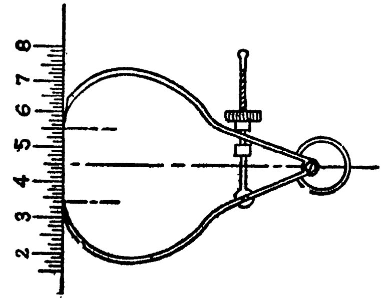
Vernier calipers are made in the standard sizes of 6, 12, 24, 36, and 48 inches, and 150, 300, 600, and 900 millimeters. The length of the jaws will range from 1¼ inches to 3½ inches, and the minimum inside measurement with the smallest caliper is ¼ of an inch or 6 millimeters.

The vernier caliper has a wide range of measurement, and the shape of the measuring anvils and their position with respect to the scale adapts this instrument to certain jobs where a micrometer, for example, could not satisfactorily be applied. It is also capable of being used for both outside and inside measurements—a feature which makes this tool one of the most versatile precision instruments in the shop. However, it does not have the accuracy of a micrometer. In any one inch of its length a vernier caliper should be accurate within .001 of an inch. In any 12 inches it should be accurate within .002, and increase about .001 for every 12 inches thereafter.

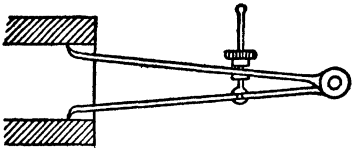
Other Types of Calipers. For outside measurements, such as the thickness of a metal plate or the diameter of a cylinder, the *outside* calipers are used as shown herewith. In using these calipers, the instrument must always be kept square with the work to be measured. For inside measurements, such as the inside diameter of a pipe or a tube, the *inside* calipers are used; when using this, the axis of the calipers must line up with the axis of the work, and the tips of the caliper legs must be square with the largest portion of the diameter being measured. In using the micrometer calipers described in the preceding paragraphs the following points should be observed:



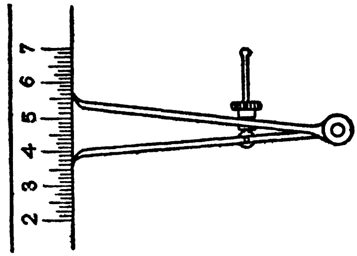
Taking a measurement with an outside calipers.



Transferring the measurement.



Taking a measurement with an inside calipers.



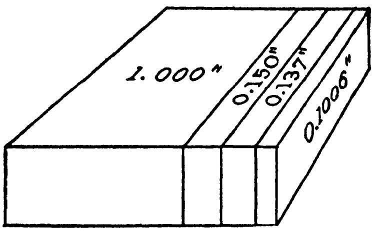
Transferring the measurement.

1. Never force the caliper by using too much pressure.
2. Always take the reading while the micrometer is held on the work.
3. Always open the micrometer before removing it from the part measured.
4. Never use the micrometer on a moving part while a machine is running.

Precision Measurements. Line-graduated measuring instruments, such as the scale, by which measurements up to .001 inch are taken, are "non-precision instruments," despite the fact that careful, accurate measurements may be made with them. But when dimensions are controlled and reproduced to thousandths of an inch or better, whether by a micrometer or by gage blocks, the measurements are known as *precision measurements.* Such precision measuring instruments themselves are calibrated by means of gage blocks, which are taken as the industrial standards of length. For ordinary shop operations, such as patternmaking, forging, stamping, rough machining, etc., precision measurements are not required. But where tolerances are very small, as for example in the manufacture of watches, clocks, delicate instruments, typewriters, firearms or parts of automotive engines, the use of precision gage blocks is indispensable.

Gage Blocks. Gage blocks are rectangular blocks of steel with a measuring surface at each end. They are made of a special alloy steel, heat treated and aged so that internal, molecular stress and strain are at a minimum, thereby decreasing the tendency of the metal to warp or "grow." The surfaces of the blocks are mechanically polished by a special process by which a number of blocks are finished at the same time to identical size. The flat surfaces of the blocks are ground and polished to an extremely high finish resembling that of burnished silver. They are the most accurate pieces of manufactured metal in the world; their errors are generally less than 5/1,000,000 of an inch per inch of length, and some of them are accurate to within 2/1,000,000 of an inch. They are probably the nearest approach of a man-made device to a perfect mathematical plane. Since a rise in temperature of 1° causes the blocks to expand $6/1,000,000$ of an inch, they are finished, and subsequently used, in a room kept at a constant temperature of 68°F. Gage blocks usually range in length from .010 of an inch up to 20 inches. They are generally obtainable in sets from 5 to as many as 85 blocks of different lengths. With a large set of over 80 blocks more than 100,000 gages in steps of .0001 of an inch may be made by using various combinations of blocks.

To combine them, the surfaces, having first been thoroughly cleaned, are slid one on the other, with a slight inward pressure; this is sometimes spoken of as "wringing" them together. When placed together in this way they stick with remarkable tenacity; when lifted in the air, blocks that have been wrung together properly have been known to support a weight of somewhat over 200 lb., although the precise reason for this amazing adhesion has never been satisfactorily explained.



Showing how gage blocks are used in combination to make up the measurement of 1.3876

As already mentioned, appropriate gage blocks are put together to secure any desired combination necessary for a particular measurement. For example, if a measurement such as 1.3876″ is desired, the following blocks would be selected: 1.000″; .150″; .137″; .1006″; their sum equals 1.3876″, which is the required measurement. In many cases it will be possible to find several combinations of blocks to give the measurement required. A complete set of blocks may contain the following sizes:

1.000″	.050″	.550″	.101″	.114″	.126″	.138″	.1001″
2.000″	.100″	.600″	.102″	.115″	.127″	.139″	.1002″
3.000″	.150″	.650″	.103″	.116″	.128″	.140″	.1003″
4.000″	.200″	.700″	.104″	.117″	.129″	.141″	.1004″
	.250″	.750″	.105″	.118″	.130″	.142″	.1005″
.010″	.300″	.800″	.106″	.119″	.131″	.143″	.1006″
.020″	.350″	.850″	.107″	.120″	.132″	.144″	.1007″
.030″	.400″	.900″	.108″	.121″	.133″	.145″	.1008″
.040″	.450″	.950″	.109″	.122″	.134″	.146″	.1009″
	.500″		.110″	.123″	.135″	.147″	
			.111″	.124″	.136″	.148″	
			.112″	.125″	.137″	.149″	
			.113″				

Exercise 14.

Using the above table of sizes, find appropriate combinations of blocks for each of the following measurements:

1.	.3944	5.	.3982	9.	1.8539
2.	.5532	6.	.4338	10.	9.6402
3.	.4265	7.	3.9061	11.	7.2944
4.	.666	8.	2.7072	12.	4.0098

Accuracy of Gage Blocks. The guaranteed accuracy of gage blocks is expressed as ±.000002 of an inch *in an inch.* In other words, a gage block measuring .500 of an inch may vary between .499998 and .500002 and still be acceptable. Furthermore, a block 4.000000 inches long may vary 4 times .000002, or .000008 of an inch, i.e., from 3.999992 to 4.000008 inches and be acceptable. It might be supposed that the accumulated error in a stack of five or six blocks might be considerable; it would be, were it not for the fact that the variations mentioned are distributed according to the laws of probability—some *plus* and some *minus* —so that they counterbalance, and the total error in a stack of blocks rarely exceeds twice that of a single block; frequently it is even *less* than that of a single block.

Uses of Gage Blocks. It should also be pointed out that gage blocks are made and used for various classes of work, i.e., various levels of accuracy. Thus a very high grade of blocks, with a range of error of from 5 to 20 millionths of an inch, would be used for inspecting tools, verifying gages, and calibrating various instruments. A second-grade set of blocks, with a range of error of from 20 to 40 millionths of an inch, might be used in layout work,—dies, jigs, fixtures, etc. A third-rate set, with errors ranging from 40 to 100 millionths of an inch, would be suitable for setting up milling, grinding, and drilling machines, or for the inspection of machine parts, etc.

Standard Sets and Working Sets. Wherever a considerable amount of precision measuring is required in an industrial plant, it is customary to use primary and secondary standards, that is, a master set of gage blocks, and a working set. The master set is carefully preserved, and is used only to check the accuracy of the working sets used in the shop. The master set is usually sent to the Bureau of Standards at Washington, or returned to the manufacturer, once a year or so for certification; each block is then checked for flatness, parallelism and length, and is certified as varying so-and-so many millionths of an inch.

As the working sets get older they become worn through use and handling. It must be remembered that all gage blocks are extremely delicate; even the natural moisture of the hands contains an acid which may stain the blocks if they are handled too much. Hence as the blocks wear out, they are progressively used for less important work. When they become so worn that the error is greater than 100 millionths of an inch they are either discarded or chromium-plated and relapped to size. In recent years, gage blocks have been made from carboloy for use in working sets because of the high resistance of carboloy to wear.

5. PERCENTAGE

Meaning of Per Cent. As we have already seen, a fractional part of any given amount may be expressed either as a common fraction or as a decimal fraction. There is still a third way: by using a per cent. A *per cent* is simply a decimal fraction written without the decimal point, with the (%) sign used instead of the decimal point to indicate the fact that the denominator is "hundredths"; thus

$$
\begin{aligned}
.28 &= 28\% \\
.125 &= 12.5\% = 12\tfrac{1}{2}\% \\
.003 &= .3\% = \tfrac{3}{10}\% \\
.0425 &= 4.25\% = 4\tfrac{1}{4}\%
\end{aligned}
$$

Per cents may be added, subtracted, multiplied, etc., just as other numbers having similar units or denominations; thus

$$
\begin{aligned}
15\% + 3\% + 10\% &= 28\% \\
100\% - 16\% &= 84\% \\
6 \times 3\tfrac{1}{2}\% &= 21\% \\
24\% \div 3 &= 8\%
\end{aligned}
$$

Finding a Per Cent of a Number. The commonest problem involving per cents is that of finding a given per cent of a given number. The given number is called the *base;* the per cent required is called the *rate;* and the result of finding the per cent (or taking the rate) is called the *percentage.* Thus, in finding 20% of 750, we say:

$$
\begin{aligned}
&20\% \text{ of } 750 = ? \\
\text{or } &.20 \times 750 = 150;
\end{aligned}
$$

here 750=base, .20=rate per cent, and 150=percentage. Finding a per cent of a number is therefore seen to be a simple matter of multiplying a number by a decimal, i.e., using the formula

$$
\begin{gathered}
Percentage = Base \times Rate \\
\text{or, } P = B \times R
\end{gathered}
$$

EXAMPLE 1: The cost of material for a job is estimated at $48; an additional 15% is allowed for miscellaneous expenses. Find (a) the amount of the allowance, and (b) the total estimated cost.

SOLUTION: (a) $48×.15=$7.20, *Ans.*

(b) $48+$7.20=$55.20, *Ans.*

EXAMPLE 2: A specimen of Monel metal consists of 68% of nickel, 28% of copper, and the remainder of small amounts of other metals. Find (a) the number of pounds of nickel and copper in a piece of Monel metal weighing 18½ lb.; (b) the number of pounds of material other than nickel and copper in this specimen.

SOLUTION: (a) 18.5×.68=12.58 lb. nickel, *Ans.*
18.5×.28= 5.18 lb. copper, *Ans.*
(b) 68%+28%=96%
100%—96%= 4%, other metals
18.5×.04 =.74 lb., *Ans.*

Exercise 15.

1. If ice is 91.7% as heavy as water, and water weighs 62.4 lb. per cu. ft., find the weight of a cubic foot of ice.
2. When tested, a gasoline engine actually gave 86% of its rated horsepower. If the engine was rated at 110 H.P., what was the actual horsepower delivered by the engine?
3. A motor is running at 2600 revolutions per minute. If the speed of the motor is increased by 6½%, how many r.p.m. will it then make?
4. If the loss in power due to friction in a certain device is 28%, what amount of power will this device transmit when supplied with 125 horsepower?
5. A machine shop casting weighs 60 lb. Due to an error in dimensioning, it is necessary to remove 12½% of the casting by machine. How many pounds of metal must the machinist remove?
6. The employees in a shop are to receive a wage increase of 12%. If junior mechanics have been getting $10.75 a day and helpers $7.80, what is the daily wage rate of each after the raise goes into effect?
7. An alloy used for bearing metal contains 14% tin, 27% antimony and 59% lead; how much of each of these metals is required to make 150 lb. of bearing metal?
8. A pattern weighs ¼ as much as the casting to be made from the pattern. If the casting weighs 110 lb., what is the weight of the pattern?
9. An inexperienced operator turned out 250 pieces of work on a stamping machine When inspected, it was found that 2½% of them had to be rejected as imperfect. How many pieces were rejected?
10. In making a certain piece to measurement, an allowance of 1½% either way is permitted. If the dimension called for is 4.8 in., what are the "outside limits" permissible?

11. In mixing a batch of concrete, about 15% of the weight is cement, 30% is sand, and 55% is gravel. If the dry mixture has a total weight of 1450 lb., how many pounds of each are used?

Changing a Per Cent to a Fraction. It is sometimes convenient to change a per cent into an equivalent common fraction. To do this, the per cent is first expressed as a decimal fraction, which in turn is then reduced to lowest terms.

EXAMPLE 1: Change 28% to a common fraction.

SOLUTION: $28\% = .28 = 28/100$

$28/100 = 14/50 = 7/25$, *Ans.*

EXAMPLE 2: Express 18¾% as a common fraction.

SOLUTION: $18\frac{3}{4}\% = .18\frac{3}{4} = .1875 = \frac{1875}{10{,}000} = 75/400 = 3/16$, *Ans.*

Changing a Common Fraction to a Per Cent. In a somewhat similar way, the above process may be reversed, and any common fraction can in turn be expressed as a per cent. Thus the numerator is first divided by the denominator, the quotient being written as a decimal; the decimal is then converted to a per cent by moving the decimal point two places to the right and annexing the % sign.

EXAMPLE 1: Change 3/8 to the per cent form

SOLUTION: $3/8 = 3 \div 8$

```
8)3.000
  ----
  .375
```

$.375 = 37.5\%$, *Ans.*

EXAMPLE 2: Express 13/17 as a per cent.

SOLUTION:

```
     .7647+
17)13.000
   11 9
   ----
    1 10
    1 02
    ----
       80
       68
       ---
       120
       119
       ---
```

or, $.7647 = 76.47\%+$, *Ans.*

EXAMPLE 3: Change $\frac{6}{5}$ to a per cent.

SOLUTION: $6 \div 5 = 1.2 = \frac{120}{100} = 120\%$, *Ans.*

Determining the Rate Per Cent. Another common problem arising in connection with the use of per cents is that of determining the rate per cent, i.e., finding what per cent one number is of another. It is precisely the same problem as that explained in the preceding paragraph. Its relation to percentage will be seen at once from the following:

$$Percentage = Base \times Rate$$

If $P = B \times R$,

then $R = \frac{P}{B}$

or $Rate = Percentage \div Base.$

EXAMPLE 1: What per cent of \$600 is \$39?

SOLUTION: $\frac{\$39}{\$600} = \frac{39}{600} = \frac{13}{200} = .065 = 6\frac{1}{2}\%$, *Ans.*

EXAMPLE 2: By changing the plan of a pattern a saving of $5\frac{1}{4}$ lb. is made in a casting originally weighing 46 lb. What is the per cent of weight thus saved?

SOLUTION: $\frac{5\frac{1}{4}}{46} =$

```
     .114+
46)5.25
   4 6
   ---
    65
    46
   ---
   190
   184
   ---
```

Saving $= .114+ = 11.4\%+$, *Ans.*

Exercise 16.

1. An automatic production machine turns out 36 pieces of work per hour. After certain adjustments had been made, the machine turned out 42 pieces per hour. What is the per cent of increase in the production rate?
2. A shop hand receiving 80¢ an hour is given an increase in wages. If he now receives 92¢ an hour, by what per cent was his wage rate increased?

3. Before a bronze casting was machined it weighed 31¼ lb.; after the machining operations had been performed it weighed 27½ lb. What was the per cent of reduction in weight?
4. A steam pressure of 175 lb. per sq. in. is increased to 220 lb. per sq. in. What is the per cent of increase?
5. If the price of gasoline is increased from 16½¢ a gallon to 17¼¢, what per cent of increase is this?
6. A power saw uses a "cutting compound" made by mixing 5 quarts of lard oil with water enough to fill a 10-gallon tank. What per cent of the compound is oil?
7. A beam is expected to support a maximum load of 40 lb. per linear foot. If it is designed to withstand a load of 56 lb. per ft., what factor of safety was allowed?
8. In making a pattern, a designer allows 3/16 in. per foot for shrinkage. What per cent is this?
9. A bottle contains 250 gm. of potassium chloride "analyzed reagent." The label states that it contains 0.083 gm. of magnesium chloride impurity. What per cent is this?
10. A carpenter added 1¼ pints of alcohol to 2½ quarts of shellac. By what per cent did he "thin" the shellac?
11. In a printing of 2500 leaflets the press operator spoiled 45 copies. What was the per cent of spoilage?
12. By tuning up a Diesel engine an operator saves an average of 15 gal. of fuel per day. If the average consumption of fuel had been 175 gal. per day, what was the per cent of saving in fuel?

Finding the Base. A less frequently occurring problem is the following: having given the percentage and the rate per cent, what was the original base? When expressed by means of the formula, the method of answering this type of question may readily be seen; thus

$$\text{since } P = B \times R,$$
$$\text{then } B = P \div R,$$

or the base is found by dividing the percentage by the rate.

EXAMPLE 1: A factory "let out" 240 of its employees. If this meant that 15% of its employees were laid off, how many were originally employed?

SOLUTION: $15\% = 240$
$1\% = 240 \div 15 = 16$
$100\% = 100 \times 16 = 1600$, *Ans.*

EXAMPLE 2: By increasing the amount of "filler" in a certain grade of paper stock the weight of the paper was increased by 10%. If the stock now weighs 16½ lb., what did it weigh originally?

SOLUTION: $100\%+10\%=110\%$
$110\%=16.5$ lb.
$1\%=16.5\div110=.15$ lb.
$100\%=100\times.15=15$ lb., *Ans.*

Exercise 17.

1. In normal times a shop produces a certain number of finished pieces per day. When production is stepped up 25% by working overtime, 39 additional pieces per day are produced. What is the daily production under normal conditions? when overtime work is done?
2. A certain material shrank, on account of moisture, 4% in size. If it now measures 21.6″, what was its original size?
3. The inventory clerk of a factory has on hand 300 ft. of round bars of a certain size. If this is 30% of the average stock of that size, how much of it is usually kept on hand?
4. The weight of zinc in a casting made of Lumen metal is 74.8 lb. If Lumen metal consists of 5% aluminum, 10% copper, and the rest zinc, find the weight of the casting.
5. The overhead in a manufacturing plant is 24% of the value of the goods produced. If in a certain month the overhead expense amounted to $38,400, what was the value of the goods produced?
6. A foreman's weekly wage is increased by 12%. If his raise amounts to $5.82 per week, what was his original weekly wage?

Practical Uses of Percentage. Many practical applications of percentage problems arise in the trades, especially in connection with the business aspects of industrial practice. Thus profit is expressed as a certain per cent of the volume of sales, or sometimes as a per cent of the cost; so also, are labor costs, overhead, and cost of materials expressed as per cents. Other items frequently expressed in per cents are the cost of maintenance of physical plant and equipment; the cost of repairs and replacements; taxes; insurance, such as fire insurance, flywheel insurance, boiler explosion, plate-glass insurance, etc., the premiums on accident policies, workmen's compensation, unemployment insurance, pensions, old-age security benefits and the like; all these are usually figured in terms of percentages.

Depreciation. All equipment, such as machinery, tools, fixtures, trucks, even buildings, decrease in value as time goes on. This decrease in value is known as *depreciation.* It is usually due to the actual wearing out of the equipment so that it is no longer serviceable, although it sometimes becomes necessary to discard equipment even before it is worn out completely because of new inventions or improved styles. Frequently the equipment to be discarded has a certain junk value, or scrap value, when it is disposed of. The number of years that the equipment remains in use is called its estimated or "useful life". The amount of annual depreciation can be computed in several ways. One of the simplest and commonest methods is to suppose that it depreciates in value in *equal amounts* each year of its life. This is not actually the case with some types of equipment. However, when this method is used, the computation is as follows:

$$\textit{Annual depreciation} = \frac{C - S}{n}$$

where C=original cost, S=scrap value, and n=number of years of estimated life. On this basis (called the "constant-value" method), the *rate of depreciation* is given by:

$$\textit{Rate of depreciation} = \frac{\textit{Annual depreciation}}{\textit{Original cost}}$$

EXAMPLE: A power drill worth \$875 when new has an estimated life of 15 years, and its scrap value is \$50. Using the constant-value method, find
(a) the annual depreciation charge, and
(b) the annual rate of depreciation.

SOLUTION: (a) Amount of annual depreciation $= \frac{\$875 - \$50}{15} = \frac{\$825}{15}$
$= \$55$, *Ans*

(b) Rate of annual depreciation $= \frac{\$55}{\$875} = .0629 = 6.29\%$, *Ans.*

Commercial Discount. When material or equipment is purchased, it is usually subject to *discount.* This means that a certain per cent of the quoted price is allowed, either for immediate cash payment, or because of a trade allowance on the list price, or because it is bought in large quantity. The following examples will illustrate such discounts.

EXAMPLE 1: A bill for lumber amounted to \$176; a discount of 2½% for cash was allowed. If prompt payment was made, what did the lumber actually cost the purchaser?

SOLUTION: $\$176 \times .025 = \$\ 4.40$, discount
$\$176 - \$4.40 = \$171.60$, net cost, *Ans.*

EXAMPLE 2: A wrench listed in the catalog at \$2.75 is subject to a discount of 33⅓%. What is the net cost?

SOLUTION: $\$2.75 \times 33\frac{1}{3}\% = \$.916 = \$.92$, discount
$\$2.75 - \$.92 = \$1.83$, net cost, *Ans.*

EXAMPLE 3: A supply house lists a rotary pump at \$75, sub'ect to a discount of 20% and 10%. Find the net price.

SOLUTION:
\$75×20%=\$15, first discount
\$75—\$15 =\$60, first "net price"
\$60×10%=\$ 6, second discount
\$60—\$6 =\$54, net price, *Ans.*

NOTE: Either discount can be computed first; the result will be the same. The two discounts *cannot be added,* however; "20% and 10%" is *not equivalent* to a 30% discount. This is the short cut, if you want one:

100%—20%=80%
100%—10%=90%
(.9)×(.8)=.72=72%
\$75×.72=\$54, *Ans.*

Exercise 18.

1. Brass fittings are offered by a manufacturer at \$7.20 a hundred. At a discount of 25%, what is the cost of 200 fittings?
2. Micrometers are quoted at \$32 a doz., less 25% and 20%. What is the net cost of one micrometer?
3. A wood plane was purchased at a net cost of \$3.80. If the catalog price was \$4.75, what per cent of discount was offered?
4. A lathe costing \$1200 has an estimated life of 15 years. If it has a scrap value of \$150, what is the annual depreciation charge?
5. If a piece of equipment costing \$240 has no scrap value and has a useful life of 8 years, what is the annual rate of depreciation?
6. A factory building cost \$40,000 to erect. If the depreciation is figured at 2% of the original cost each year, what is the "book value" of the building after 22 years?
7. A solution of a solid in water contains 24% of the solid by weight. How much of this solid is dissolved in 15 oz. of the solution?
8. As calculated, the speed of a pulley is 300 revolutions per minute. When checked with a tachometer, however, it actually made only 280 r.p.m., due to belt slippage. What is the per cent of slippage?
9. Ordinary air contains about 19.5% of oxygen by volume. How many cubic feet of oxygen are there in 2000 cu. ft. of air?
10. Because of a leaky valve, 2½ quarts of oil were lost out of every 20 gallons. What per cent was lost?
11. A factory regularly employing 320 men increases its force by 15%. How many men are now working in this factory?
12. The diameter of a rod when measured by a micrometer is found to be 1.23 in. If the blueprint called for a diameter of 1.18 in., what is the per cent of error?

13. If the working hours are increased from 40 hours per week to a 44-hour week, what is the increase in a weekly pay roll amounting to $5600, if the same hourly wage rates are maintained?
14. An upholsterer allows 10% extra on the cost of material for nails, thread, glue, sandpaper, etc. If the wood and fabric for an upholstered stool come to $4.38, what is the total estimated cost of material for 2 doz. such stools?
15. When making a bid on an installation job, a contractor allows 30% of the estimated cost for additional "overhead." If he offers a bid of $156, what did he estimate the cost to be?

6. RATIO AND PROPORTION

Ratio as a Comparison. A *ratio* is a device for comparing two quantities of the same kind. For example, if two strips of metal are 8 in. and 10 in. long, respectively, we could say that the second is 2 in. longer than the first, or 25% longer than the first. This is a *difference* method of comparing them; telling how much more or less. Another way of comparing them would be to say that one is $\frac{4}{5}$ as long as the other, or the second is $\frac{5}{4}$, i.e., $1\frac{1}{4}$, times as long as the first. This is the *ratio* method of comparison: telling how many times as much. We say that the lengths of the boards are "in the ratio of 4 to 5, or 5 to 4," which may be written as 4:5, or 5:4. A ratio, then, is simply a *fraction* which gives the comparison at a glance; the above ratio might also be written as $\frac{4}{5}$ or $\frac{5}{4}$ instead of "4:5" or "5:4"; in fact, the colon (:) is really an abbreviation for "÷" with the horizontal line omitted. Notice that a ratio is independent of the units of measure; i.e., the two lengths mentioned above are in the ratio of 4:5 whether we express them in inches, feet, or yards. The units "cancel out," and the ratio remains 4:5. When comparing two quantities by the ratio method, however, care should be taken that the numbers to be compared are always expressed in the same units of measure to begin with.

EXAMPLE 1: A spindle is 3″ high and $\frac{3}{4}$″ in diameter. What is the ratio of its diameter to its height?

SOLUTION: $\frac{3}{4} \div 3 = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4} = 1:4$, *Ans.*

EXAMPLE 2: A rectangular sheet of tin measures 12′6″ in length by 8′4″ in width. Find the ratio of the length to the width.

SOLUTION: $12'6'' = 12\frac{1}{2}$ ft.
$8'4'' = 8\frac{1}{3}$ ft.
$12\frac{1}{2} \div 8\frac{1}{3} = \frac{25}{2} \times \frac{3}{25} = \frac{3}{2} = 3:2$, *Ans.*

Exercise 19.

1. Two ladders are 12 ft. and 18 ft. long. What is the ratio of their lengths?
2. What is the ratio of the lengths of a 6″-pocket rule and a yardstick?
3. Two near-by office buildings are 24 stories and 36 stories high. Assuming that the "stories" in each building are the same height, what is the ratio of the heights of the buildings?
4. A certain style "legal size" envelope measures 4″×9½″. What is the ratio of its width to its length?
5. A photographic print is 3¼″×4¼″. What is the ratio of its dimensions?
6. A rectangle is said to have the most pleasing appearance when the ratio of its width to its length is 0.7. According to this standard, what should be the width of a rectangular placard that is 25 in. long?
7. A boy is 3 ft. 9 in. tall, and his father stands 5 ft. 9 in. Find the ratio of their heights.
8. A drawing of a flower in a biology textbook is 5¼ in. high. If the caption under the drawing reads "¾ actual size," what is the actual height of the flower?
9. The micro-photograph of a textile fibre is 2.4 cm. long. If the magnification is 1:60, how long is the actual specimen?
10. A mechanic constructed a miniature model of a machine part which was actually 3 ft. 6″ long. If he used a scale of "1 inch=½ foot," how long did he make the model?

Using Ratios. Ratios are very useful, and can be employed in many ways, as the following illustrative problems will show.

EXAMPLE 1: A board is 16 ft. long; if it is to be divided into two pieces in the ratio of 3:5, how long should each piece be?

SOLUTION: 3+5=8

one piece =⅜ of entire length, or 6 ft.
other piece=⅝ of entire length, or 10 ft., *Ans.*

EXAMPLE 2: Muntz metal consists of 6 parts of copper and 4 parts of zinc by weight. How many pounds of each metal are there in a block of Muntz metal weighing 72 lb.?

SOLUTION: 6+4=10
Ratio of copper to Muntz=6:10=.6
Ratio of zinc to Muntz =4:10=.4
72×.6=43.2 lb. copper,
72×.4=28.8 lb. zinc, *Ans.*

EXAMPLE 3: The ratio of the diagonal of a square to the side of the square is 1.4. Find (a) the diagonal of a square whose side is 30 inches; (b) the side of a square whose diagonal is 28 inches.

SOLUTION: A ratio of 1.4 is the same as 14:10, or (1.4):(1).

(a) $\text{diagonal} = 1.4 \times \text{side}$

$= 1.4 \times 30 = 42$ in., *Ans.*

(b) $\text{side} = \text{diagonal} \div 1.4$

$= 28 \div 1.4 = 20$ in., *Ans.*

Since a ratio is always a fraction, ratios are frequently expressed as per cents. The specific gravity of a substance is the ratio of its weight to the weight of an equal volume of water. Thus ether, being only about 70% as heavy as water, has a specific gravity of 0.7; ice, 0.92; air, 0.0013; aluminum, 2.6; lead, 11.37; etc.

Exercise 20.

1. One inch is equivalent to 2.5 cm. What is the ratio of an inch to a centimeter? of a centimeter to an inch?
2. One liter is equivalent to 1.06 quarts. What is the ratio of a quart to a liter?
3. Divide a 42″-rod into two pieces in the ratio of 5:7.
4. The angles of a triangle are in the ratio of 1:2:3. If the sum of the three angles equals 180°, how large is each angle?
5. One quart equals approximately .95 liters. How many liters of acid are there in a 5-gallon acid carboy?
6. The ratio of the altitude of an equilateral triangle to its side is .866. What is the altitude of such a triangle if its side is 20 inches? What is the length of the side if the altitude is 2.598 inches?
7. The smaller of two connected pulleys makes 180 revolutions per minute while the larger one makes 45 revolutions. What is the ratio of their speeds? If the smaller one is speeded up to 220 r.p.m., what will be the speed of the larger, assuming the same speed ratio?
8. Aluminum metal expands .000013 of its length per Fahrenheit degree rise in temperature. If the original length of an aluminum bar is 200 cm., what is its length when raised 100°F?
9. Monel metal consists of 68½% of nickel, 1½% of iron, and the rest, copper. How many pounds of copper are there in a Monel metal casting weighing 60 lb.?
10. A ton of ready-mix concrete consists of cement, sand and gravel in the ratio of 1½:3½:5. How many pounds of each ingredient are there in the mixture?
11. If a sample of petroleum weighs 55 lb. per cu. ft., and water weighs 62.5 lb. per cu. ft., find the specific gravity of the petroleum.

12. If the specific gravity of ice is 0.92, what is the weight of 8 cu. ft. of ice, assuming that water weighs 62.5 lb. per cu. ft.
13. Brazing metal is an alloy made up of 20% zinc and 80% copper. What is the ratio of zinc to copper?
14. German silver (white metal) consists of 2 parts zinc, 3 parts nickel, and 5 parts copper. Find (a) the ratio of zinc to copper; (b) of copper to nickel; (c) what per cent of the alloy is nickel?
15. A commonly used mixture for concrete is made up of 1 part of cement, $2\frac{1}{2}$ parts of sand, and 4 parts of stone. Find (a) the ratio of sand to stone; (b) the ratio of cement to sand; (c) what per cent of the concrete mixture is sand?

Scale Drawings. In representing distances on a map or dimensions on a plan or blueprint, it is necessary to use a scale, or to "scale down" the quantities, all in the same ratio. Thus on a given map, an inch might represent 300 miles, in which case two cities located $2\frac{1}{2}$ inches apart on the map would actually be 750 miles distant from each other. Or the floor plan of a house might be drawn to a scale of $1'' = 10$ ft.; in that case a room which on the plan is $\frac{3}{4}''$ wide is actually $7\frac{1}{2}$ ft. wide, and a room 18 ft. long would be represented by a line 1.8 in. long.

EXAMPLE 1: A catalog picture of a machine part is labeled as being "$\frac{2}{5}$ actual size." If the length of the part in the picture is 3.8 in., what is its actual length?

SOLUTION: $3.8 \times \frac{5}{2} = 9.5$ in., *Ans.*

EXAMPLE 2: The working model of a machine is to be on a scale of 1:50. If a connecting piece of this machine is actually 16′8″ long, how long should the corresponding piece of the model be made?

SOLUTION: $16'8'' = 200''$

$200'' \times \frac{1}{50} = 4''$, *Ans.*

EXAMPLE 3: On a blueprint the scale used is $1'' = 10'$. Find (a) the actual size of a distance of $2\frac{1}{4}''$ on the blueprint; (b) how long on the blueprint an actual distance of 36 ft. ought to be.

SOLUTION: $1'' = 10$ ft.

(a) $2\frac{1}{4}'' = 10 \times 2\frac{1}{4} = 22.5$ ft., *Ans.*

(b) 1 ft. $= \frac{1}{10}$ in.

36 ft. $= 36 \times \frac{1}{10} = 3.6$ in., *Ans.*

Exercise 21.

1. The dimensions of the top of a rectangular workbench are 4′ by $8\frac{1}{2}'$. What should be its dimensions on a scale drawing, if the scale is $\frac{1}{4}'' = 1'$?

2. A living room is 20′×14′. On the architect's plan, what are its dimensions, if the scale used is ⅛″=1′?
3. If the scale used in the following is 1″=1′, fill in the missing values:

Actual Length	2′	5′	8′	20′		6½′
Scale Length	2″	5″			15″	

4. If the scale used in the following is ½″=10′, fill in the missing values:

Actual Length		20′	5′	24′		
Scale Length	⅛″				2″	3⅝″

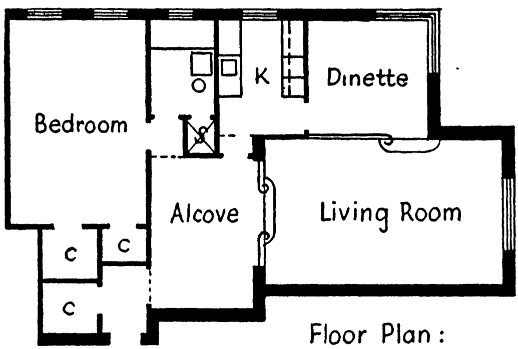
5. Complete the following table:

Scale Used	*Actual length*	*Scale length*	*Actual length*	*Scale length*
(a) 1″=20 ft.	35 ft.	?	?	3¼″
(b) 1″= 6″	1½ ft.	?	?	10″
(c) ⅛″= 1 ft.	12 ft.	?	?	2⅜″
(d) ½″= 1 mile	25 mi.	?	?	5¾″
(e) ¼″=10 ft.	45 ft.	?	?	4½″

6. The floor space of a storage bin is 14 ft. by 24½ ft. Using a scale of 1″=1′, what are the dimensions of the floor space on the architect's plan?
7. The dimensions of a metal plate for a machine are 6½ ft.×9 ft. What should its dimensions be on a blueprint, if the scale used is ½″=1′?
8. The scale of miles on a map is 1″=150 mi. How far apart on the map are two cities that are actually 1225 miles from each other?
9. On the plan of an apartment house a bedroom is shown, measuring 3½″ by 5¾″. If the floor plan is drawn to a scale of ¼″=1′, what are the actual dimensions of the room?
10. The detailed plan of a working model is represented on a draftsman's drawing by a scale of 1″=6″. What are the dimensions of a rectangular part measuring 4½″×6¼″ on the drawing?
11. A surveyor's map is drawn on a scale of 1″=10 feet. How far is it actually from one point to another that is 3¾ inches from it on the map?
12. On the layout of a camp site 1″=0.5 mile. It is 1¼ miles from the mess hall to a certain cottage. How far apart are these two places on the diagram?
13. Complete the blank spaces:

Length of object	1′	5′	6″		5″	4 yd.
Length of drawing	1″	½″		3′		6″
Scale used			½	¼	1′=6″	

14. A carpenter is using a blueprint with a scale of ½″=1′. What are the dimensions of a door that is 1¾″×3½″ on the blueprint?
15. Find the dimensions of (a) the living room, (b) the dinette, (c) the bedroom, and (d) the alcove.



Floor Plan:
3½ Room Apartment
Scale 1/16″ = 1′ 0″

Proportion. The word "proportion" is one which is often used carelessly, or with only a vague notion of what is really meant. Strictly speaking, when we compare two quantities, we cannot speak of their "proportion"; we can refer to the ratio between them, or what part or what per cent one is of the other. When we speak of a proportion, we have in mind *four* quantities, and are comparing them *in pairs,* as ratios. In other words, if two ratios are equal to each other, they are said to form a proportion. Thus, 3:5=12:20 is a proportion. Putting it another way, if the ratio between any two quantities is numerically equal to the ratio between two other quantities, then the four quantities are in proportion. For example, a nickel bears the same ratio to a dime that a half-dollar does to a dollar, since

$$\frac{5}{10}=\frac{50}{100}, \qquad \text{or } \frac{1}{2}=\frac{1}{2}$$

Notice that in the first case, all four quantities are expressed in terms of the same units, *viz.,* cents, although the units don't appear; in the second case, the two ratios are reduced to lowest terms to show their equality. Or again, if a man 6 ft. tall casts a shadow 8 ft., then a pole 18 ft. high will cast a shadow 24 ft. long; or, 6:8=18:24.

$$\frac{6}{8}=\frac{18}{24} \qquad \frac{3}{4}=\frac{3}{4}$$

In other words, a proportion is simply an equation stating that two fractions are equal. For example,

$$\text{if } 2:3=10:15, \qquad \text{then } \tfrac{2}{3}=\tfrac{10}{15},$$

or $2\times15=3\times10$, which might be called "cross-multiplying" the four *terms* of the proportion. If only three of the four terms of a proportion are known, the remaining term can easily be found by the principle of "cross-multiplication," as shown below.

EXAMPLES:

1. If $\frac{2}{3}=8/n$, find the missing number "n."
   Multiplying "crisscross,"
   $2\times n=8\times3$
   $2n=24$
   $n=12$
   Check: $\frac{2}{3}=\frac{8}{12}$

2. If $\frac{4}{5}=n/20$, find the missing quantity "n."
   Cross-multiplying:
   $4\times20=5\times n$
   $5n=80$
   $n=16$
   Check: $\frac{4}{5}=\frac{16}{20}$

3. If $3/n=\frac{5}{8}$, find n.
   Multiplying:
   $3\times8=5\times n$
   $5n=24$
   $n=\frac{24}{5}=4\frac{4}{5}$
   Check: $3\div4\frac{4}{5}=\frac{5}{8}$

4. If $n/4=\frac{2}{7}$, find n.
   $7\times n=2\times4$
   $7n=8$
   $n=\frac{8}{7}=1\frac{1}{7}$
   Check: $1\frac{1}{7}\div4=\frac{2}{7}$

Exercise 22.

Find the missing term in each of the following proportions:

1. $\frac{3}{4}=15/n$	4. $\frac{5}{6}=n/24$	7. $6/n=\frac{2}{3}$	10. $n/40=\frac{3}{5}$
2. $\frac{2}{5}=16/n$	5. $\frac{3}{8}=n/32$	8. $7/n=\frac{1}{6}$	11. $n/180=\frac{24}{120}$
3. $\frac{2}{3}=14/n$	6. $\frac{9}{4}=n/12$	9. $n/16=\frac{3}{8}$	12. $88/n=3\frac{1}{7}$

Direct Proportion. A proportion in which the ratios vary in the same order is called a *direct proportion.* For example, the volume of a gas (under constant pressure) *varies directly* with the temperature: as the temperature increases, the volume increases; as the temperature decreases, the volume decreases. This may be expressed mathematically as follows:

$$\frac{V_1}{V_2}=\frac{T_1}{T_2};$$

note that the "subscripts" of the letters are in the same order.

Example 1: If 8 bolts weigh 10 oz., how much will 48 similar bolts weigh?

Solution: Let x represent the required weight.

$$8/48 = 10/x$$
$$8x = 10 \times 48$$
$$x = \frac{10 \times 48}{8} = 60 \text{ oz., } \textit{Ans.}$$

Example 2: The elongation of a certain spring varies directly with the weight applied. If a weight of 48 oz. causes an elongation of 2 inches, (a) what will the elongation be when a weight of 60 oz. is applied? (b) what weight will produce an elongation of 1½ inches?

Solution:

$$\frac{e_1}{e_2} = \frac{w_1}{w_2}$$

(a)
$$2/x = 48/60$$
$$48x = 2 \times 60$$
$$x = \frac{2 \times 60}{48} = 2\tfrac{1}{2} \text{ in., } \textit{Ans.}$$

(b)
$$\frac{2}{1\tfrac{1}{2}} = \frac{48}{x}$$
$$2x = \tfrac{3}{2} \times 48$$
$$x = \frac{\tfrac{3}{2} \times 48}{2} = 36 \text{ oz., } \textit{Ans.}$$

Exercise 23.

1. If 100 pages of a book measure ⅜″ in thickness, what will be the thickness of a book of 480 pages of the same quality of paper?
2. If 7½ gallons of paint cost $10.50, what will 40 gallons of paint cost?
3. Eight stamping machines turn out 560 pieces of work in one hour. How many pieces will 5 of these machines turn out in an hour and a half?
4. If a bomber flies 840 miles in 3 hours, how far will it fly in 8½ hours at the same rate?
5. If the electrical resistance of 250 ft. of a certain wire is 150 ohms, how many ohms resistance will 875 ft. of the same wire have?
6. Metal castings are often sold by the pound. If a casting weighing 240 lb. costs $12.80, what is the weight of a similar casting that costs $41.60?

7. A section of a steel girder 18 ft. in length weighs 450 lb. How long is another section of the same girder if it weighs 1050 lb.? What is the weight of a piece 10 ft. long?
8. The volume of a gas under constant pressure varies directly as the absolute temperature. If $V=420$ when $T=225$, what is the value of V when $T=175$? For what value of T will V equal 1200?
9. At 75 lb. pressure per sq. in., a certain exhaust pipe discharges 270 cu. ft. of gas per minute. Assuming direct variation, how many cu. ft. are discharged at 80 lb. pressure? What pressure is required to discharge 450 cu. ft. per minute?
10. The distance traveled by sound varies directly as the time required to hear the sound. A storm is 1½ miles away, and the sound of the thunder reaches an observer 7.2 seconds after the lightning flash is seen. Some time later the thunder is heard 4.5 seconds after the flash. How much nearer is the storm the second time?

Inverse Proportion. A proportion in which the ratios vary in the opposite order is called an *inverse proportion.* For example, the volume of a gas (under constant temperature) *varies inversely* as the pressure: as the pressure increases, the volume *decreases,* and as the pressure decreases, the volume *increases.* This may be expressed mathematically as follows:

$$\frac{V_1}{V_2}=\frac{P_2}{P_1};$$

note that in this case the subscripts of the letters are in *reverse* order.

EXAMPLE: The current (C) in an electric circuit varies inversely as the resistance (R). If $C=2$ amperes when $R=55$ ohms, find (a) the current when $R=220$; (b) find the resistance R when $C=5$ amperes.

SOLUTION: $\frac{C_1}{C_2}=\frac{R_2}{R_1}$

(a) $\frac{2}{C_2}=\frac{220}{55}$

$220\ C_2=2\times 55$

$C_2=110\div 220=\frac{1}{2}$ ampere, *Ans.*

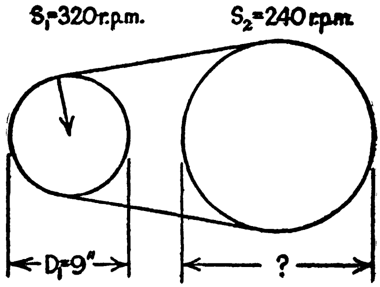
(b) $\frac{2}{5}=\frac{R_2}{55}$

$5R_2=2\times 55$

$R_2=110\div 5=22$ ohms, *Ans.*

Inverse proportion is well illustrated by the relation of the diameters of pulleys and gears. Whenever two pulleys having different diameters are connected, the smaller pulley always rotates more times than the larger. Likewise, when two gears with different diameters are in mesh, the smaller (having the lesser number of teeth) turns more rapidly than the larger one.

EXAMPLE 1: Two pulleys are connected with a belt; the smaller, having a diameter of 9″, makes 320 revolutions per minute, while the larger makes 240 r.p.m. Find the diameter of the larger pulley.



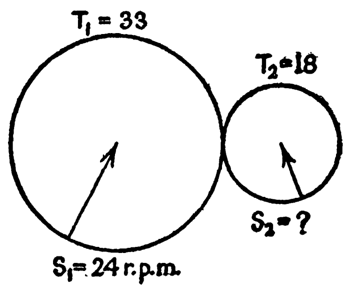
SOLUTION: $\frac{D_1}{D_2}=\frac{S_2}{S_1}$

$\frac{9}{D_2}=\frac{240}{320}$

$240\ D_2=9\times320$

$D_2=16''$, *Ans.*

EXAMPLE 2: Two gears in mesh have 33 teeth and 18 teeth. When the larger makes 24 r.p.m., how many r.p.m. does the smaller gear make?



SOLUTION: $\frac{T_1}{T_2}=\frac{S_2}{S_1}$

$\frac{33}{18}=\frac{S_2}{24}$

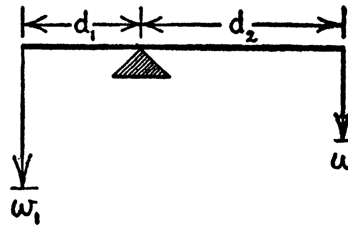
$18\ S_2=24\times33$

$S_2=44$ r.p.m., *Ans.*

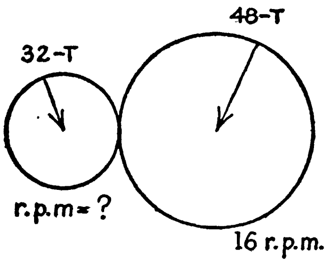
Exercise 24.

1. The resistance (R) in an electric circuit varies inversely as the current (C). If $R_1=40$ and $C_1=54$, find C_2 when $R_2=180$; find R_2 when $C_2=36$.
2. The volume of a gas at constant temperature varies inversely as its pressure. If the volume of the gas in a certain cylinder is 720 cu. in. at a pressure of 20 lb. per sq. in., what will be its volume under a pressure of 25 lb. per sq. in.? What pressure will be required to reduce its volume to 200 cu. in.?

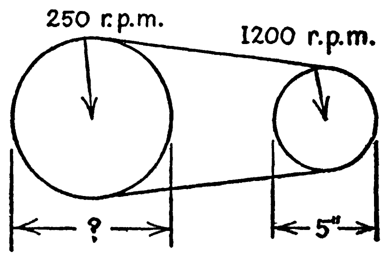
3. The so-called principle of moments, or balanced turning tendencies, follows the law of inverse proportion; for $\frac{w_1}{w_2}=\frac{d_2}{d_1}$, or $w_1d_1=w_2d_2$. Find w_2 if $w_1=4$ lb., $d_1=6$ in., and $d_2=8$ in.; find d_2 if $w_1=24$ lb., $w_2=30$ lb., and $d_1=10$ in.



4. The density of a substance varies inversely as its volume when the mass (weight) is constant, or $D_1 : D_2=V_2 : V_1$. If the density of a gas is .0015 when its volume is 8000 cu. cm., what will its density be when it has been compressed to a volume of 5000 cu. cm.?

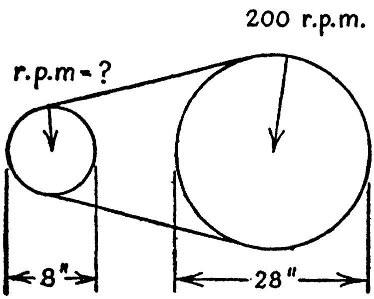


Ex. 5

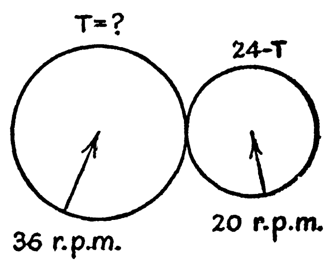


Ex. 6

5. Find the number of r.p.m. of the smaller gear.
6. Find the diameter of the larger pulley.



Ex. 7



Ex. 8

7. Find the speed of the smaller pulley.
8. Find, to the nearest whole number, the number of teeth in the larger gear.
9. If a driving pulley has a diameter of 20″ and its speed is 750 r.p.m., what is the speed of a 6″ driven pulley?

10. A circular buffer 6″ in diameter should revolve at 1800 r.p.m. If it is driven by a line shaft revolving at 400 r.p.m., find the diameter of the driving pulley on the line shaft that will be needed to obtain the desired speed of the buffer.

CHAPTER II

ELEMENTS OF ALGEBRA

William L. Schaaf

7. NUMBERS AND SYMBOLS

The Language of Algebra. The methods of algebra are essentially an extension of arithmetic. In other words, the numbers, symbols and operations used in algebra are the same as those used in arithmetic, only they are *more general* in character. This means (1) that *letters* as well as numbers are used to represent quantities, and (2) that numbers are regarded as having *quality* as well as quantity.

Consider the use of letters as symbols of quantity. In arithmetic, we say a force of 10 lb.; in algebra, we say a force of F lb., and we think of the letter F as having *various numerical values,* either one after another, or even "all at once." Or again, in arithmetic we say a bolt has a diameter of 1½″; in the language of algebra we say it has a diameter of d inches; etc.

The following will illustrate how *verbal statements* are translated into *algebraic symbols:*

Verbal statement	*Algebraic formulation*
(1) Four times a number.	$4n$
(2) Sum of two numbers decreased by a third number.	$a+b-c$
(3) Three times a number increased by twice a second number.	$3k+2m$
(4) Sum of two numbers divided by 5 times a third number.	$\frac{x+y}{5z}$
(5) Twice the product of two numbers increased by ⅔ of a third number.	$2ab+\frac{2}{3}c$
(6) Ten times a number decreased by six.	$10n-6$

Exercise 25.

"Translate" into words each of the following algebraic expressions:

1. $2a+3b$
2. $5x-\frac{1}{2}y$
3. $\frac{a-b+c}{3}$
4. $\frac{2}{3}k+8$
5. $P-\frac{1}{N}$
6. $5Rh$
7. $\frac{m-n}{2p}$
8. $\frac{9}{5}C+32$
9. $2lw+2lh+2wh$
10. $\frac{H-h}{T-t}$

Substitution. The process of finding the numerical value of an algebraic expression for certain specific values of the letters that occur in the expression is known as *substitution.*

EXAMPLE 1: Find the value of $\frac{\pi D}{4}h$, when $h=12$, $D=3.5$, and $\pi=3\frac{1}{7}$.

SOLUTION: $\frac{1}{4} \cdot \frac{22}{7} \cdot \frac{7}{2} \cdot 12=33$, *Ans.*

EXAMPLE 2: Find the value of $2ak-\frac{3}{4}m$, when $a=20$, $k=2.5$, and $m=1.6$.

SOLUTION: $(2)(20)(\frac{5}{2})-(\frac{3}{4})(1.6)=100-1.2=98.8$, *Ans.*

Exercise 26.

Find the numerical value of each of the following:

1. $2x+3y$, when $x=4\frac{1}{2}$ and $y=0.8$.
2. $\frac{3}{4}h-2$, when $h=16.44$.
3. $\frac{p+q}{3r}$, when $p=10.3$, $q=8.6$, and $r=.07$.
4. $2\frac{1}{4}D+1\frac{3}{4}$, when $D=.816$.
5. $\frac{4}{3}\pi R^3$, when $\pi=3\frac{1}{7}$ and $R=3.5$.
6. $0.3707p-0.0052$, when $p=\frac{1}{5}$.
7. $\frac{9}{5}C+32$, when $C=99.5$.

Complete each of the following tables of values as indicated:

(8) D	$D/2$	(9) k	$8k$	(10) H	$.75H$	(11) N	$\frac{1}{8}N$
4	?	0		4		0	
6	?	1		8		2	
8	?	2		10		4	
10	?	3		12		6	
15	?	4		24		10	

(12)

R	$\frac{4}{3}R$
2	
4	
6	
8	
12	

(13)

p	$\frac{2}{3}p$
1	
3	
5	
7	
10	

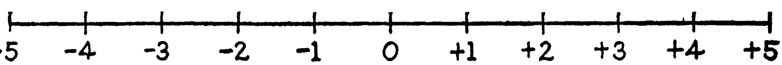
(14)

h	$\frac{h}{3.14}$
0	
1	
2	
10	

(15)

x	$4\frac{1}{2}x$
0	
1	
2	
4	
10	

Positive and Negative Quantities. The second feature which distinguishes algebra from arithmetic is the use of *negative* numbers as well as positive numbers. Accordingly, all numbers and letters are assumed to be either positive or negative (except zero), and are designated, respectively, as $+$ or $-$; thus $+5$, -2, $+\frac{1}{2}$, -0.15, etc. The essential significance of these signs is to denote *oppositeness,* i.e., oppositeness of direction on a number scale, as suggested below:



Addition of signed numbers in algebra is equivalent to combining "steps" or "intervals" along a number scale, the signs of the numbers indicating the "direction" from the zero point taken in each step. The "sum" of an algebraic addition is thus the *net result* of combining two or more such steps. For example, by algebraic addition, we get:

$+3$	$+4$	-5	-3	$+2$	-1	$+4$
$+2$	-3	$+2$	-2	-3	$+3$	-4
$+5$	$+1$	-3	-5	-1	$+2$	0

If no sign appears before a quantity, it is assumed to be positive.

RULE I: *To add two signed numbers whose signs are* alike, *find their arithmetic sum and prefix the same sign that both have.*

RULE II: *To add two signed numbers whose signs are* opposite, *find their arithmetic difference, and then prefix the sign of the (numerically) larger quantity.*

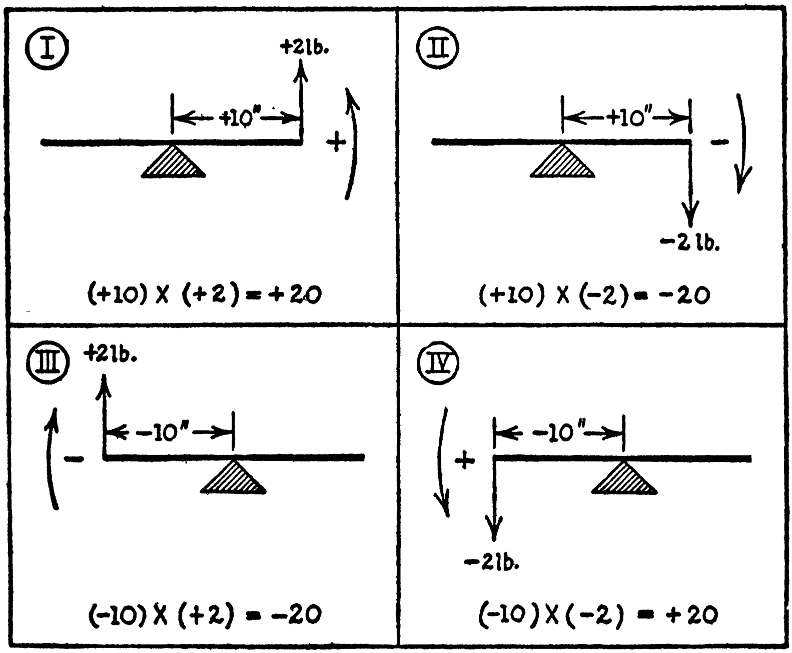
Subtraction being the opposite or inverse process of addition, it might be expected that we also reverse something when we subtract in algebra; we do. To subtract a quantity in algebra, we add the same quantity *with the opposite sign.* Hence

RULE III: *To subtract one signed number from another, change the sign of the subtrahend; then add them algebraically.*

For example, when subtracting the lower quantity from the upper in each case, we get:

+6	+4	—7	—8	+ 3	— 5	—6
+4	—3	+2	—8	— 7	+ 9	—3
+2	+7	—9	0	+10	—14	—3

Multiplication and Division with Signed Numbers. The rules concerning signs when multiplying and dividing signed numbers are simple, and, while they may seem a bit arbitrary, their reasonableness may be seen by studying the accompanying diagrams of a lever (or see-saw), where it



is agreed that distance to the right of P are +, and to the left, —; where it is also agreed that downward pulls are negative (—) and upward pushes positive (+); and finally, that clockwise turning is negative and counterclockwise, positive.

RULE IV: *In multiplication, like signs give a positive product, and unlike signs give a negative product.*

For division, the same rule holds true: if the signs of the dividend and divisor (or numerator and denominator) are alike, the quotient is positive; if unlike, the quotient is negative.

Exercise 27.

1. Add:

$+14$	$+22$	$+17$	-31	-12	$+15$	-18
$-\ 8$	$+16$	-17	$+12$	-10	-27	30

2. Subtract:

$+15$	$+18$	-16	$+32$	-14	-18	-25
$-\ 8$	$+\ 5$	-10	$+32$	-12	18	-25

3. Multiply:

$(+3)(-21)$	$(+16)(+\frac{1}{8})$	$(-10m)\times(-3)$
$(-6)(-10)$	$(-\frac{1}{2})(-12.4)$	$(4)\times(-3a)$
$(-15)(+4)$	$(-6)\times(12k)$	$(-8)\times(6r)$

4. Divide:

$(+18)\div(-2)$	$(+24)\div(+2)$	$(+12)\div(-6)$
$(-27)\div(+3)$	$(+50)\div(-25)$	$24\div(-3a)$
$(-15)\div(-5)$	$(-16)\div(-16)$	$(-48x)\div 6$

Addition and Subtraction of Similar Terms. In algebra, quantities to be added or subtracted are known as *terms*. *Similar terms* are terms similarly composed of the same letters; thus, for example, $3a$ and $5a$ are similar terms; so are $3xy$, $-2xy$ and $6xy$. *Only similar terms can be added and subtracted.* The sum of $4n$, $-3n$, and $5n$ is $6n$; but $4n$ and $3p$ cannot actually be added (until numerical values are substituted for both n and p). In terms like these the number before a letter (or group of letters mutiplied together) is known as the numerical coefficient of the term.

RULE V: *To add (or subtract) similar terms, add (or subtract) their numerical coefficients, and annex the same letters to the new coefficient.*

Exercise 28.

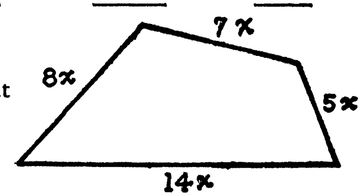
1. Add:

$15p$	$+28r$	$-\ 3m$	$+16.4h$	$-\ 8\frac{1}{2}x$
$6p$	$-\ 2r$	$12m$	$-\ 7.8h$	$-14x$

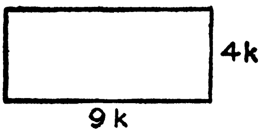
2. Subtract:

$12k$	$-20a$	$+15xy$	$-18lw$	$+\frac{1}{2}ab$
$8k$	$-14a$	$-11xy$	$+18lw$	$-2ab$

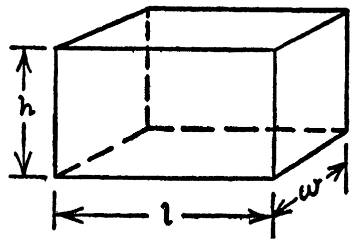
3. Find the perimeter of the figure at the right.

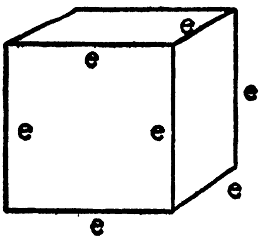


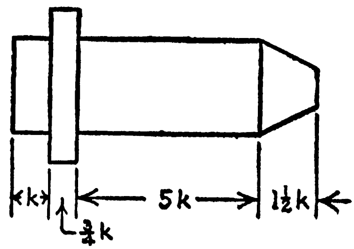
4. What is the perimeter of the rectangle shown below.



5. Find the sum of the twelve edges of the rectangular solid shown below.



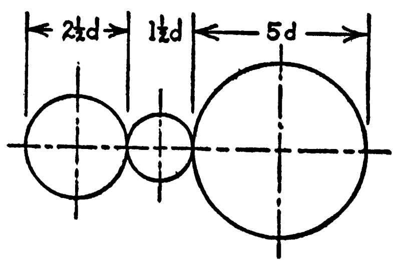




6. What is the sum of all the edges of the cube?

7. What is the overall length of the piece of work shown?

8. Find the overall diameter of this train of gears.



Multiplying and Dividing Literal Numbers. Although only similar terms can be added and subtracted, the operations of multiplication and division can be performed on any number of terms, whether similar or not. Thus, $2a \times 3b = 6ab$; $5mh \times 3r = 15mhr$. Or again, $12a \div 4a = 3$; $16xy \div 8x = 2y$; etc. Furthermore, to show how similar terms are multiplied and divided, study the following examples:

$$a \times a = a^2$$
$$b \times b \times b = b^3$$
$$3a \times 5a = 15a^2$$
$$4p \times 3pq = 12p^2q$$
$$5h^2r \times 2hr^3 = 10h^3r^4$$

$$\frac{6a^2}{2a} = 3a$$
$$\frac{-12m^2x}{2mx^3} = \frac{-6m}{x^2}$$
$$\frac{-8a^2b^3}{-4ab^2} = 2ab$$

NOTE: In algebra, the multiplication sign ($\times$) is frequently omitted between two letters to be multiplied together; instead, a dot is placed between them, or nothing at all, to indicate multiplication. Thus

$$(2m)\times(r)=2m\cdot r=2mr.$$

Exercise 29.

1. Find the product of:

$8\times 4m$	$3\frac{1}{7}\times 21h$	$4a^2b\cdot 3ab^2$
$10\times 3k$	$\frac{2}{3}s\cdot\frac{1}{2}t$	$5x^3\cdot x^4$
$5p\times 3q$	$\frac{1}{3}r^2\cdot 15h$	$m^2\cdot m^3$
$6h\times 2ab$	$4R^2\cdot\frac{1}{3}R$	$12xy\cdot\frac{1}{2}xy$
$4a^2\times 3ab$	$3.8at\times 2at^2$	$(\frac{1}{2}a^3)\cdot(\frac{1}{3}a^2)$

2. Divide:

$14m\div 2$	$240k\div 32$	$15r^2\div 5rs^3$
$32x\div 4x$	$35y^2\div 5y$	$-8cd^3\div c^2d$
$24p\div p$	$12a^3\div ab$	$2\pi R^2h\div\pi R$
$15\div 5x$	$20xy\div 4xy$	$-10a^3b^6\div 5ab^4$

Terms and Factors. The distinction about to be made between *terms* and *factors* is very important. Thus in the expression $(2)(a+b)$, one factor is "2" and the other is the *entire* quantity "$a+b$." *Factors* are quantities which are to be multiplied or divided; *terms* are quantities that are to be added or subtracted. Thus, in the expression $2a+2b$ there are *two terms,* viz., "$2a$" and "$2b$"; each of these terms consists of two factors, to wit, 2 and a, and 2 and b, respectively. In the first instance the parenthesis ( ) indicates that the quantity enclosed within it is to be regarded as *all one quantity* so far as the operations of multiplication (or division) are applied to it. In short, whatever the "a" is multiplied by, the "b" must also be multiplied by. Or

$$3(a+b)=3a+3b;$$
$$\tfrac{1}{2}(p+q-m)=\tfrac{1}{2}p+\tfrac{1}{2}q-\tfrac{1}{2}m;\ \text{etc.}$$

Exercise 30.

Write each of the following without the parentheses:

1. $2(l+w)$
2. $k(a+b-c)$
3. $a+(n-1)d$
4. $I(R+r)$
5. $P(1+RT)$
6. $S=\frac{n}{2}(a+l)$
7. $\frac{h}{2}(B+b)$
8. $\pi(R^2-r^2)$
9. $L(t_0-t_1)$
10. $2\pi r(l+r)$
11. $\frac{n}{2}[2a+(n-1)d]$
12. $V=\frac{h}{6}(b_1+b_2+4M)$

Factor each of the following expressions:

13. $p+pi$
14. $hp-hq$
15. I^2R+I^2r
16. $2ab-2ac$
17. $P+Prt$
18. $\frac{1}{2}B_1h+\frac{1}{2}B_2h$
19. $\pi Dl+\pi D$
20. $2\pi Rl+2\pi R^2$

Find the numerical value of each of the following:

21. $\frac{5}{9}(F-32)$, when $F=212$; when $F=68$.
22. $\frac{1}{2}h(B_1+B_2)$, when $h=6\frac{1}{2}$, $B_1=10.4$, and $B_2=5.6$.
23. $\frac{12(D-d)}{L}$, when $D=8.5$, $d=6.25$, and $L=48$.
24. $2\pi h(R^2-r^2)$, when $R=10$, $r=6$, $h=21$, and $\pi=3\frac{1}{7}$.

Squaring a Number. In order to find the area of a square we simply multiply the length of the side *by itself.* This is called squaring a number. The square, or second power, of a number is the product of the number multiplied by itself (not multiplied by 2); thus $3^2=3\times3=9$, or $10^2=10\times10=100$. If an indicated product contains two equal factors, it is the "square" of either one of them.

The Cube of a Number. Similarly, if a number is multiplied *by itself 3 times* (not multiplied by 3), the product is called the *cube* of the number; this is suggested by the fact that the volume of a cube equals the edge multiplied by itself three times. An indicated product having three equal factors is called the "cube" of any one of them. Thus, $2^3=2\times2\times2=8$; $5^3=5\times5\times5=125$; $10^3=10\times10\times10=1000$; etc.

Exercise 31.

Find the value of each of the following:

1. $8^2=?$
2. $16^2=?$
3. $6^3=?$
4. $12^3=?$
5. $1^3=?$
6. $3^3=?$
7. $20^2=?$
8. $7^3=?$
9. $(\frac{1}{2})^2=?$
10. $(\frac{2}{3})^2=?$
11. $(\frac{1}{4})^3=?$
12. $(\frac{1}{10})^2=?$
13. $(\frac{3}{4})^3=?$
14. $(.01)^2=?$
15. $(.01)^3=?$
16. $(.001)^2=?$

Other Powers. A number may be "raised to any power" desired. Thus:

$3^4=3\times3\times3\times3=81$ ("fourth power of 3")
$2^5=2\times2\times2\times2\times2=32$ ("fifth power of 2")
$10^6=10\times10\times10\times10\times10\times10=1{,}000{,}000$ ("sixth power of 10")

As a matter of fact, using various "powers" of 10 is a convenient device employed by scientists and engineers. Let us first study the following table of powers of the base 10:

$10^1=10$	$10^{-1}=0.1$
$10^2=100$	$10^{-2}=0.01$
$10^3=1000$	$10^{-3}=0.001$
$10^4=10,000$	$10^{-4}=0.0001$
$10^5=100,000$	$10^{-5}=0.00001$
$10^6=1,000,000$, etc.	$10^{-6}=0.000001$, etc.

Using these values, we can "abbreviate" a number like 290,000 as 29×10^4, since $29\times10^4=29\times10,000=290,000$. Similarly, a large number like 3,920,000,000 may be expressed more conveniently as 3.92×10^9, since $10^9=1,000,000,000$. Very small numbers, like .00000057 may be written as 57×10^{-8}; and so on.

Exercise 32.

Express each of the following in full:

1.	34×10^5	4.	43×10^{10}	7.	8.62×10^8	10.	23×10^{-8}
2.	62×10^8	5.	5.2×10^6	8.	24.63×10^{12}	11.	4.9×10^{-9}
3.	91×10^9	6.	13.8×10^{11}	9.	35×10^{-6}	12.	32.6×10^{-11}

Express each of the following as a power of 10:

13. 37,000,000	15. 12,400,000	17. 0.00004	19. 0.000392
14. 5,800,000	16. 4,900,000,000	18. 0.00000028	20. 0.00000000076

21. Astronomers use a unit known as the "light year," which is the distance traveled by light in one year. If this distance is equal to 5.8825×10^{12} miles, express this distance without using a power of 10.
22. Scientists have measured the wave length of sodium light and found it to be 0.0005893 millimeters. Express this in abbreviated standard form.

Laws of Exponents. When a quantity, whether a number or a letter, is raised to a power, the small number which indicates the power to which it is to be raised, and which is written to the upper right of it, is called an *exponent*. Thus in 25^2, the exponent is *2;* in $3x^5$, the exponent is *5*. Letters, too, may be used as exponents; thus, in 10^m, $4a^x$, and P^{k+1}, the exponents are *m, x,* and $(k+1)$, *respectively.* When powers are to be combined by multiplication or division, the following principles must be observed:

Principle	*Illustration*
(1) $a^m \cdot a^n=a^{m+n}$	(1) $x^4 \cdot x^3=x^{4+3}=x^7$
(2) $a^m\div a^n=a^{m-n}$	(2) $y^5\div y^2=y^{5-2}=y^3$
(3) $(a^m)^n=a^{mn}$	(3) $(p^3)^2=p^{(3)\,(2)}=p^6$
(4) $(ab)^m=a^mb^n$	(4) $(ar)^4=a^4r^4$

Meaning of a Root. The operation which is the inverse (opposite) of raising to a power is called "taking the root." In other words, finding the square root of a quantity means finding the two equal factors which multiplied give that quantity; to find a cube root means finding the three equal factors of that product; etc.

EXAMPLES

Raising to a Power	*Expressing the Root*
$a^2=a\times a=P$	$\sqrt{P}=a$
$x^3=x\,.\,x\,.\,x=Q$	$\sqrt[3]{Q}=x$
$k^5=k\,.\,k\,.\,k\,.\,k\,.\,k=R$	$\sqrt[5]{R}=k$

Roots may also be expressed by using *fractional exponents,* as shown:

$$\sqrt{m}=m^{1/2} \qquad \sqrt[4]{x}=x^{1/4}$$

$$\sqrt[3]{a}=a^{1/3} \qquad \sqrt[5]{p}=p^{1/5}$$

$$\sqrt[3]{y^2}=y^{2/3}$$

$$\sqrt{A^3}=A^{3/2}$$

Negative Exponents. We have already seen that $10^{-1}=1/10$ or 0.1; $10^{-2}=1/100$, or 0.01; etc. A quantity with a *negative exponent* indicates that *that quantity with the same positive exponent is to be divided into one,* i.e., it indicates the *reciprocal* of that power; thus

$$a^{-1}=\frac{1}{a};\quad x^{-2}=\frac{1}{x^2};\quad p^{-5/3}=\frac{1}{\sqrt[3]{p^5}}$$

This may be better understood by studying the following summary:

$a^5=aaaaa$	$a^{-1}=\frac{1}{a}$
$a^4=aaaa$	$a^{-2}=\frac{1}{aa}=\frac{1}{a^2}$
$a^3=aaa$	$a^{-3}=\frac{1}{aaa}=\frac{1}{a^3}$
$a^2=aa$	$a^{-4}=\frac{1}{aaaa}=\frac{1}{a^4}$
$a^1=a$	$a^{-5}=\frac{1}{aaaaa}=\frac{1}{a^5}$

The reason why a negative exponent causes a factor to appear in the denominator may be seen by applying principle No. 2 above. Thus

$$\frac{a^3}{a^5}=a^{3-5}=a^{-2};$$

$$\text{but } \frac{a^3}{a^5}=\frac{\overset{1\,1\,1}{\not a\not a\not a}}{\not a\not a\not a aa}=\frac{1}{a^2};$$

$$\text{therefore } a^{-2}=\frac{1}{a^2}.$$

$$\text{Similarly, } \frac{x}{x^4}=x^{1-4}=x^{-3};$$

$$\text{but } \frac{x}{x^4}=\frac{\overset{1}{\not x}}{\not x xxx}=\frac{1}{x^3};$$

$$\text{therefore } x^{-3}=\frac{1}{x^3}.$$

The same principle also shows why any quantity with a *zero exponent* must always be equal to 1, no matter what the quantity itself may be; thus

$$\frac{a^5}{a^5}=a^{5-5}=a^0;\ \text{but } \frac{a^5}{a^5}=1;\ \text{hence } a^0=1$$

$$\frac{x^{13}}{x^{13}}=x^{13-13}=x^0;\ \text{but } \frac{x^{13}}{x^{13}}=1;\ \text{hence } x^0=1.$$

Roots and Powers of Fractions. Just to make sure, the reader is reminded that in raising a fraction to a power, both numerator and denominator are raised to that power; thus

$$(2/3)^2=\frac{2^2}{3^2}=4/9;\ (3/10)^3=.027;\ \left(\frac{2\pi R^2}{h}\right)^2=\frac{4\pi^2R^4}{h^2}.$$

Likewise when taking a root:

$$\sqrt{\frac{9}{16}}=\frac{\sqrt{9}}{\sqrt{16}}=\frac{3}{4};\ \sqrt[3]{\frac{1}{8}}=\frac{1}{2};\ \sqrt{\frac{2E}{m^2}}=\frac{\sqrt{2E}}{m}$$

Exercise 33.

1. Write the following without using exponents:
$a^{1/3}$; $x^{1/4}$; $(5a)^{1/2}$; $p^{2/3}$; $A^{3/2}$; $(mn)^{1/3}$.

2. Write the following without using radicals:

$$\sqrt{p};\quad \sqrt[4]{k};\quad \sqrt[3]{a^2};\quad \sqrt[5]{10x};\quad \sqrt{\frac{2s}{g}};\quad \sqrt{\frac{A}{\pi}}.$$

3. Find the value of the following:

$x^{1/2}$, when $x=144$.
$k^{1/3}$, when $k=27/8$.
$a^{2/3}$, when $a=8$.
$x^{2/3}$, when $x=8$.
$p^{1/2}$, when $p=36a^2$.
$3a^2b^3$, when $a=2$ and $b=1$.
$2x^3y^2$, when $x=3$ and $y=2$.
$(2x^3)^2$, when $x=10$.

4. What is the numerical value of each of the following?

$25^{-1/2}$	$(.04)^{1/2}$	$(.027)^{1/3}$	10^{-3}
$4^{3/2}$	$64^{2/3}$	$16^{3/4}$	$(5x)^{-2}$
$27^{2/3}$	$32^{1/5}$	$8^{-1/3}$	$(16a^4)^{-1/2}$

8. FORMULAS AND EQUATIONS

Meaning of a Formula. Many of the computations used in shop problems are either simplified or more clearly understood by the use of formulas. A *formula* is simply a mathematical statement of a principle or a rule describing the relation between two or more quantities. This mathematical statement shows that there is an *equality* between certain quantities; in other words, the formula translates a verbal rule into algebraic symbols. Thus a formula is very similar to an *equation*. For example, since there are 12 inches to every foot, this verbal rule can be stated as a formula by writing $I=12F$; or, since the percentage equals the base multiplied by the rate, we have the formula $P=BR$; or again, if in the lever, one weight multiplied by its distance from the support *balances* (i.e., equals) the other weight multiplied by *its* distance, then we can write this as a formula by saying: $w_1d_1=w_2d_2$.

Evaluating Formulas. It will be seen that a formula may involve two, three, four or even more quantities. If the formula expressing the relation or connection between these quantities is known, and if a particular value is known for every quantity in the formula *except one*, then the value of that remaining one is easily found. This is sometimes called "evaluating a formula," or "substituting in a formula."

Example 1: If $P=2(l+w)$, find P when $l=7.9$ and $w=5.2$.

Solution: $P=2(7.9+5.2)$
$=2(13.1)=26.2$, *Ans.*

Example 2: If $A = \frac{1}{2}h(B_1 + B_2)$, what is the value of A when $h=8$, $B_1 = 4\frac{1}{2}$, and $B_2 = 6\frac{1}{2}$?

Solution: $A = \frac{1}{2}(8)(4\frac{1}{2} + 6\frac{1}{2})$
$= \frac{1}{2}(8)(11) = 44$, *Ans.*

Example 3: If $S = \frac{1}{2}at^2$, find S when $a=32.2$ and $t=3$; also when $a=980$ and $t=10$.

Solution: $S = \frac{1}{2}(32.2)\ (3)^2$
$= \frac{1}{2}(32.2)\ (9) = 144.9$, *Ans.*
$S = \frac{1}{2}(980)\ (10)^2 = 49{,}000$, *Ans.*

Example 4: If $C = \frac{5}{9}(F-32)$, find C when $F=212$; when $F=32$.

Solution: $C = \frac{5}{9}(212-32) = \frac{5}{9}\ (180) = 100$, *Ans.*
$C = \frac{5}{9}(32-32) = \frac{5}{9}\ (0) = 0$, *Ans.*

Exercise 34.

1. If $V = lwh$, find V when $l = 14''$, $w = 6\frac{1}{2}''$ and $h = 4''$.
2. If $A = 6e^2$, find A when $e = 2\frac{1}{2}''$.
3. In the formula $V = v_0 + gt$, find the value of V when $v_0 = 200$, $g = 32$, and $t = 8$.
4. Given the relation $F = \frac{9}{5}C + 32$, what is the value of F when $C = 18^0$?
5. A formula for the amount of money due on a loan at simple interest is: $A = P\left(1 + \frac{nr}{12}\right)$. Find A when $P = \$500$, $r = .04$, and $n = 6$ months.
6. An electric current (I) is expressed by the formula $I = \frac{E}{R+r}$; find I when $R = 85$, $r = 3$, and $E = 220$.
7. Under certain conditions the energy of a moving body is given by $E = \frac{Mv^2}{2g}$; find E when $M = 2000$, $v = 80$, and $g = 32$.
8. If $I = \frac{\pi abC^2}{6}$, what is the value of I when $a = 3.6$, $b = 2.1$, $C = 2$, and $\pi = 3\frac{1}{7}$?
9. In designing modern automobile highways, the following formula relating to motor trucks is sometimes used: $W = k(L+40)$, where W = total gross weight of truck with its load, L = distance in feet between first and last axles of the truck, or truck and trailer. If the value of k in a certain state = 720, and $L = 42$ ft., find the value of W.
10. The heat generated in an electric circuit is expressed by the formula $H = 0.24Ri^2t$. If R, the resistance, equals 55 ohms; i, the current, equals 2 amperes; and t, the time, equals $\frac{1}{2}$ hour, find the amount of heat (H) produced (expressed in calories).

11. A formula sometimes used for finding the horse power rating of a gas engine is: $\text{H.P.}=\frac{D^2n}{2.5}$, where D=diameter of the cylinder in inches and n=number of cylinders, provided the piston speed is 1000 ft. per min. Find the H.P. of an engine having 12 cylinders with a 4″-bore, running at 2000 ft. per min.
12. Marine engineers find the "wetted surface" of a ship by using the formula $A=15\sqrt{Dl}$, where D=displacement in tons and l=length of the ship in feet. If a ship 900 ft. long displaces 81,000 tons, find the wetted surface A (in sq. ft.).

Simple Equations. As already stated, an equation is very similar to a formula. The chief difference, as far as we are concerned here, is this: in a formula there are at least two "quantities," or literal numbers, whereas in a simple equation there is only one such literal number. Thus:

Formulas	*Equations*
$A=lw$	$12x=5-3x$
$V=\frac{1}{3}Bh$	$\frac{3}{5}x=42$
$\frac{1}{p}+\frac{1}{q}=\frac{2}{R}$	$\frac{x}{3}+\frac{x}{2}=12\frac{1}{2}$

It turns out, therefore, for reasons that will become clearer as we go along, that in a simple equation there is *only one* numerical value which will hold true for the literal number; in the case of a formula, however, there are indefinitely many combinations of values for the literal quantities which will make the formula hold true. The process of finding the particular value which holds true for the literal quantity in any equation is called "solving the equation"; that numerical value of the letter is called the "solution," and it is said to "satisfy" the equation. For example, if it is known that $6x-17=4x+13$, it is possible to find a value which *satisfies* this equation; the desired value is 15. For if $x=15$, then substituting 15 for x in the equation we obtain:

$$(6)(15)-17=(4)(15)+13$$
$$90-17=60+13$$
$$73\equiv73$$

This shows that the equality expressed by the original equation really holds true when x has the numerical value 15, since 73 is identical with 73, or both sides of the equation have been shown to be identical.

Solving an Equation by Division. Certain types of equations are readily solved by dividing both sides of the equation by an appropriate number.

EXAMPLE 1: Solve the equation $6x=51$.

SOLUTION: Dividing each "side" of the equation by 6 we obtain

$$x=\frac{51}{6}$$

or $x=8\frac{1}{2}$, *Ans.*

EXAMPLE 2: Solve for y: $3\frac{1}{2}y=28$

SOLUTION: Dividing both sides by $3\frac{1}{2}$, we obtain

$$y=\frac{28}{3\frac{1}{2}}$$

or $y=(28)(\frac{2}{7})$

$y=8$, *Ans.*

Solving an Equation by Multiplication. Sometimes an equation may be conveniently solved by multiplication instead; each side of an equation may be multiplied by any number, provided it is the *same* number.

EXAMPLE 1: Solve: $\frac{x}{14}=3$

SOLUTION: Multiplying both sides by 14 we obtain

$x=(14)(3)$

or $x=42$, *Ans.*

EXAMPLE 2: Solve for n the equation $\frac{n}{.6}=25$

SOLUTION: Multiplying both sides by .6 we obtain

$n=(25)(.6)$

or $n=15$, *Ans.*

Exercise 35.

Solve each of the following equations for the "unknown" letter:

1. $5n=65$
2. $\frac{x}{4}=8$
3. $100x=35$
4. $\frac{1}{8}p=2.5$
5. $84=7k$
6. $\frac{y}{9}=1\frac{2}{3}$
7. $3\frac{1}{2}x=49$
8. $.05k=16$
9. $\frac{n}{.4}=30$
10. $\frac{2x}{3}=24$
11. $.3x=4.8$
12. $\frac{3}{4}h=120$

Changing the Subject of a Formula by Multiplication or Division. The same procedure as used for equations can also be applied to solving a formula for any particular letter desired. The other letters are simply regarded as numbers. This is called "changing the subject of the formula," or solving for a particular letter "in terms of the other letters."

EXAMPLE 1: Solve the formula $F=ma$ for m.

SOLUTION: Dividing both sides by a:

$$\frac{F}{a}=m, \text{ or } m=\frac{F}{a}, \textit{ Ans.}$$

EXAMPLE 2: In the formula $\frac{E}{I}=nR$, express I in terms of E, n and R.

SOLUTION: Multiplying both sides by I:

$$E=InR$$

Dividing both sides by nR:

$$\frac{E}{nR}=I, \text{ or } I=\frac{E}{nR}, \textit{ Ans.}$$

Exercise 36.

Solve each of the following formulas for the quantity specified:

1. $A=lw$; solve for l.
2. $D=\frac{M}{V}$; solve for M.
3. $D=RT$; solve for T.
4. $I=\frac{E}{R}$; solve for R.
5. $C=2\pi R$; solve for R.
6. $d=1.4s$; solve for s.
7. $PV=kT$; solve for P.
8. $I=PRT$; solve for T.
9. $A=\frac{1}{2}bh$; solve for h.
10. $S=2\pi rh$; solve for r.
11. $l_1 w_1=l_2 w_2$; solve for w_2
12. $S=\frac{1}{2}gt^2$; solve for t^2.

Solving an Equation by Addition or Subtraction. Frequently an equation is of such a form that its solution may be effected by adding or subtracting appropriate numbers, as shown below. Any number may be added to, or subtracted from, both sides of an equation; but it must be the same number.

EXAMPLE 1: Solve for x: $x+10=17$

SOLUTION: Subtracting:

$$\begin{array}{r} x+10=17 \\ 10=10 \\ \hline x \quad = 7, \textit{ Ans.} \end{array}$$

EXAMPLE 2: Solve for k: $k-6\frac{1}{2}=12$

SOLUTION: Adding:

$$\begin{array}{r} k-6\frac{1}{2}=12 \\ 6\frac{1}{2}=6\frac{1}{2} \\ \hline k \quad =18\frac{1}{2}, \textit{ Ans.} \end{array}$$

Exercise 37.

Solve each of the following equations for the unknown letter:

1. $n+12=34$
2. $30=16+k$
3. $x-25=13$
4. $60=y-35$
5. $28+p=42$
6. $h-16=20$
7. $14=x+10\frac{1}{2}$
8. $n-3.5=7.2$
9. $y+6.4=8\frac{1}{2}$
10. $x-3\frac{1}{2}=5\frac{3}{4}$
11. $18=x-4\frac{1}{2}$
12. $40.2=78.5-x$

Solving Any Kind of a Simple Equation. By "any kind" of a simple equation we simply mean an equation in which several operations may be necessary to find the solution. The procedure is illustrated by the following:

EXAMPLE: Solve: $5x+8=2x+20$

SOLUTION:
$$\begin{aligned} 5x+8-2x&=20 \\ 3x+8&=20 \\ 3x&=20-8 \\ 3x&=12 \\ x&=4, \textit{ Ans.} \end{aligned}$$

NOTE 1: These steps can be shortened by "transposing" terms; this means that any term of an equation can be "brought" from either side to the other side, *provided its sign is reversed.* Thus, in one step, we could write:

$$\begin{aligned} 5x-2x&=20-8 \\ \text{then} \quad 3x&=12 \\ x&=4 \end{aligned}$$

NOTE 2: The solution of an equation should always be checked by substituting the value obtained for the letter *in the original equation;* this should yield an *identity*. Thus:

$$\begin{aligned} (5)(4)+8&=(2)(4)+20 \\ 20+8&=8+20 \\ 28&\equiv 28, \text{ Check.} \end{aligned}$$

Changing the Subject of Any Simple Formula in General. The same procedure described in the foregoing paragraph applies to simple formulas as well as to simple equations.

EXAMPLE 1: Solve for F the temperature formula: $C=\frac{5}{9}(F-32)$.

SOLUTION:
$$\begin{aligned} C&=(\tfrac{5}{9})(F-32) \\ \tfrac{9}{5}C&=F-32 \\ F-32&=\tfrac{9}{5}C \\ F&=\tfrac{9}{5}C+32, \textit{ Ans.} \end{aligned}$$

EXAMPLE 2: Solve for R: $C=\dfrac{E}{R+nr}$

SOLUTION: $C=\dfrac{E}{R+nr}$

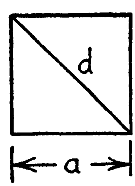
$$C\,(R+nr)=E$$
$$CR+Cnr=E$$
$$CR=E-Cnr$$
$$R=\frac{E-Cnr}{C},\ \textit{Ans.}$$

Exercise 38.

Solve the following equations:

1. $8.5x+5=22$; find x.
2. $28-4y=3(2y-4)$; find y.
3. $n=p+kr$; solve for r.
4. $C=4p+q$; solve for p.
5. $P=2(l+w)$; solve for w.
6. $V=E+Ir$; solve for I.
7. $T=\dfrac{12(D-d)}{L}$; solve for d.
8. $V=v_0+gt$; solve for t.
9. $A=\frac{1}{2}h(B+b)$; solve for B.
10. $A=P+PRT$; solve for R.
11. $l=a+(n-1)d$; solve for d.
12. $C=\dfrac{N-n}{2P}$; solve for n.
13. $D=\dfrac{2\pi R}{L}$; solve for R.
14. $N=\dfrac{S-W}{W}$; solve for S; also for W.
15. $F=\dfrac{Wv^2}{gR}$; solve for W; for R.
16. $Q=\dfrac{w(H-h)}{t_1-t_0}$; solve for H.

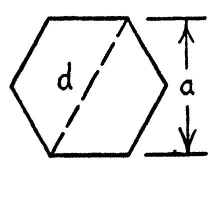
Practical Use of Formulas in the Shop. It should be pointed out that formulas of all kinds are constantly used in the various trades, in the shop, and in industrial work. For illustrative purposes as well as for reference and for self-practice a number of typical formulas are given below; these and many others will be found from time to time throughout the rest of the book.



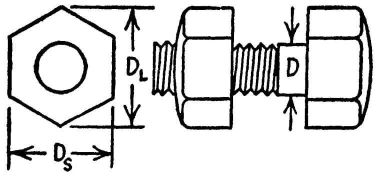
(A) *Nuts and Bolts.*

(1) Diameter of blank for square bolt:
$$d=1.414a$$

(2) Diameter of blank for hexagonal bolt:
$$d=1.155a$$



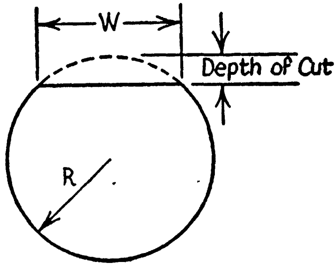
(3) Bolt and nut dimensions:
$D_S = \frac{3}{2}\ D + \frac{1}{8}$
$D_L = \frac{7}{4}\ D + \frac{1}{8}$



(B) *Depth of Cut.*

(4) When milling flats on round stock:

Depth $= R - \sqrt{R^2 - \frac{1}{4}W^2}$,
where R=radius of round bar, and W=width of flat surface.

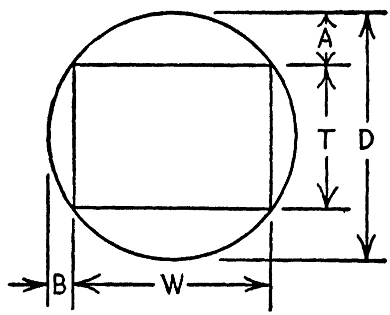


(5) When machining round stock with rectangular bars:

Depth $A = \dfrac{D - T}{2}$,

Depth $B = \dfrac{D - W}{2}$,

where D=diameter of round stock, W=width of rectangular bar, T= thickness of rectangular bar, and A and B=depths of cut.



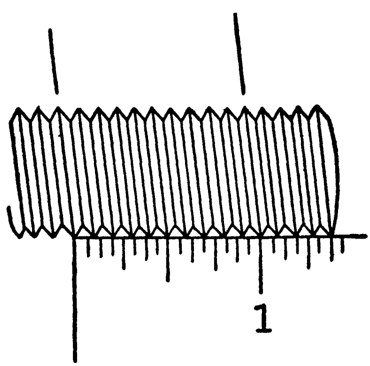
(C) *Screw Threads.*

(6) Pitch of screw (P) and number of threads per inch (N):

$$P = \frac{1}{N};\ N = \frac{1}{P}$$

(7) Depth (d) of sharp V-thread:

$$d = .866p = \frac{.866}{N}$$



Screw Thread : 10 Threads per inch

(8) Depth of American National thread:

$$d=.6495p=\frac{.6495}{N}$$

(9) Tap drill for sharp V-thread:

$$\text{Tap drill size } S=T-\frac{1.733}{N}$$

(10) Tap drill for American National thread:

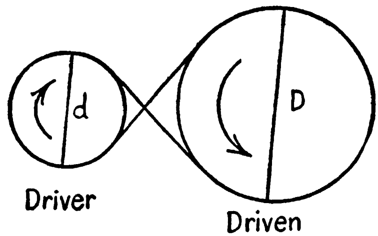
$$\text{Tap drill size } S=D-\frac{1}{N}$$

(D) *Pulley and Gear Speeds.*

(11) Speeds of belt-driven pulleys (s, S=speed, r.p.m.):

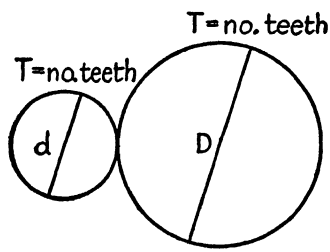
$$s=\frac{D\times S}{d}$$

$$d=\frac{D\times S}{s}$$



(12) Speed of gears in mesh (t, T= no. of teeth):

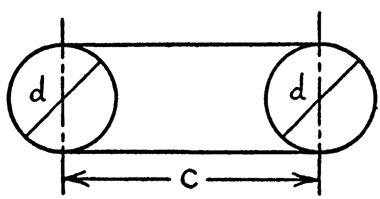
$$S=\frac{t\times s}{T}$$



(E) *Belting.*

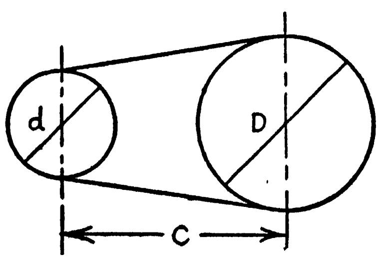
(13) Length of open belt (equal pulleys):

$$L=\pi d+2c$$



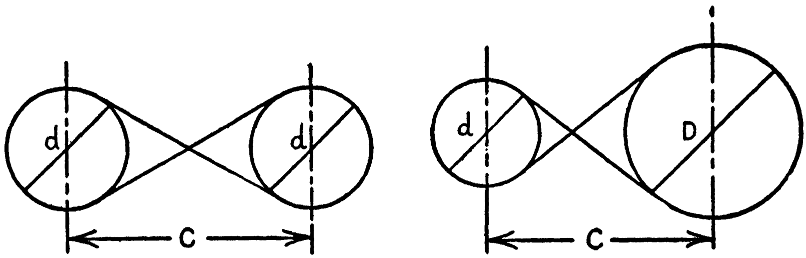
(14) Length of open belt (unequal pulleys):

$$L=\frac{\pi}{2}(D+d)+2\sqrt{c^2+(R-r)^2}$$



(15) Length of crossed belt, (equal or unequal pulleys):

$$L=\frac{\pi}{2}(D+d)+2\sqrt{c^2+(R+r)^2}$$



(16) Horsepower transmitted by belts:

$$\text{H.P.}=\frac{S\times W\times T}{33{,}000}$$, where

S =speed of belt (ft./min.)
W=width of belt (inches)
T =tension in belt (40 or 70 lb.)

(F) *Energy and Power.*

(17) Work=Force×Time:

$$W=FT$$

(18) Power=Work÷Time:

$$P=\frac{W}{T}$$

(19) Horsepower:

$$\text{H.P.}=\frac{\text{ft.-lb. per min.}}{33{,}000}=\frac{\text{ft.-lb. per sec.}}{550}$$

(20) Kinetic energy of moving object:

$$\text{K.E.}=\frac{wv^2}{2g}$$, where

w=weight (lb.)
v =velocity (ft. per min.)
g =acceleration of gravity (32 ft./sec./sec.)

(21) Indicated horsepower of steam engine:

$$\text{I.H.P.}=\frac{PLAN}{33{,}000}$$, where

P =mean effective pressure (lb./sq. in.)
L =length of stroke (ft.)
A=area of piston (sq. in.)
N=no. of strokes (2×r.p.m.)

(22) Horsepower rating of gas engine:

$$\text{H.P.} = \frac{D^2N}{2.5}, \text{ where}$$

D=diameter of cylinders (in.)
N=number of cylinders
(assumed piston speed=1000 ft. per min.)

9. SQUARE ROOT

Use of Square Root. For some reason or other, learning to find the square root of a number seems to be a stumbling block for most folks. As Stephen Leacock puts it, square root is as "obdurate as a hardwood stump in a pasture—nothing but years of effort can extract it. You can't hurry the process." Yet there is nothing really difficult about it. And it is a very practical matter, for, as we shall soon see, many problems in mensuration require the determination of the square root of a number. Finding a square root is simply the opposite of squaring a number; i.e., given the product of any number multiplied by itself, to find the original number. Several methods for extracting the square root of a number are available:

1. By inspection and approximation.
2. By using a table.
3. By the algebraic rule.
4. By means of logarithms.
5. By using the slide rule.

The last two methods will be explained in subsequent sections of the present chapter; the first three will now be discussed. Which method you use will depend largely upon the accuracy required and the availability of tables or a slide rule.

Approximation Methods. A fundamental method based upon successive trials and approximations is the following; it is somewhat laborious and therefore not particularly convenient, but will be found useful if not required too frequently, and if no other method is available. It has the advantage, however, of being simple to understand. Suppose we wish to find the square root of 44. Obviously, the value of $\sqrt{44}$ must be greater than 6, but less than 7, since 44 lies between 36 ($=6^2$) and 49 ($=7^2$). Suppose we *guess* the value to be 6.6, since 44 is somewhat closer to 49 than it is to 36. Now by actual multiplication we find that $(6.6)^2=$ 43.56, which is already quite close to 44; in fact, so close that it is safe to say that, correct to the nearest tenth, the value of $\sqrt{44}=6.6$ (the actual value=6.633). To make sure, we find by multiplication once more that $(6.7)^2=$ 44.89, which is too large by more than 43.56, is too small;

hence 6.6 is correct to the nearest tenth. If we wish to find the value correct to the nearest hundredth, we would continue the trial guesses and approximations until we were certain of the second decimal place. To be sure, the labor becomes a bit cumbersome; fortunately, for many purposes the result to one decimal place suffices. For greater accuracy, other methods are frankly superior. If the original guess of 6.6 had been less fortunate, say 6.5 or 6.7, the amount of trial multiplication needed to determine the first decimal place is not prohibitive. This method is not at all suited, obviously, for three-place figures or greater, e.g., to find $\sqrt{273}$ or $\sqrt{4136}$.

Another approximation method is based on the algebraic formula:

$$\sqrt{a^2+b}=a+\frac{b}{2a};$$

applying it to the same problem, we regard $\sqrt{44}$ as equal to $\sqrt{36+8}$, so that $a=6$ and $b=8$. Therefore $\sqrt{44}=\sqrt{6+8/12}=6\frac{2}{3}=6.67$, which is also fairly close, although when "rounded off" it would give 6.7 for the answer. This method is fairly useful for numbers somewhat larger than two-place figures. It has the advantage, moreover, of not requiring a number of tedious multiplications.

Using a Table of Square Roots. The values of squares and square roots of many numbers have been carefully worked out and are usually available in the form of *tables* in most handbooks and reference manuals. When such tables are accessible, their use is doubtless the most convenient method of finding the desired square root, unless, of course, it is required to find the value to more decimal places than given in the table; in that case, either the "algebraic" method or logarithms should be used. A convenient table of square roots is given on pages 89–93.

Using this table we find the $\sqrt{44}$ directly to be 6.633, as already mentioned. However, it should be noted that the table can be used for many more numbers than appear in the column under the heading "N". Thus $\sqrt{44.8}=6.693$; also the $\sqrt{4400}=66.33$; and $\sqrt{4480}=66.93$. In other words, moving the decimal point *two places* in N means that it must be moved *one place* in the root. Furthermore, to find $\sqrt{4.4}$, we look on page 89 under "N" for 4.4 and find that the required value of $\sqrt{4.4}=2.098$; similarly, $\sqrt{4.45}=2.110$; $\sqrt{445}=21.10$; and $\sqrt{440}=20.98$, while $\sqrt{44{,}000}=209.8$.

Square Root by the Algebraic Rule. This method is a "rule of thumb" procedure based on the algebraic relation that $(a+b)^2=a^2+2ab+b^2$. Without stopping to explain the theory in full, we shall simply illustrate the procedure.

Example 1: Find the $\sqrt{44}$.

Solution:

```
            44.000000 )6.633+, Ans.
            36
[12]  6      | 8 00
             | 7 56
[132]  3       | 4400
               | 3969
[1326]  3        | 43100
                 | 39789
```

Point off in blocks of two, beginning at the decimal point.

Largest square in $44=36; \sqrt{36}=6$, first digit in root.

Double $6=12$; "trial divisor" $=126$. Multiplying 126 by $6=756$.

Double root so far obtained; twice $66=132$. Second "trial divisor" $=1323$. Multiply 1323 by $3=3969$.

Double root so far obtained; twice $663=1326$. Third "trial divisor" $=13263$; $13263\times 3=39789$.

And so on, to as many decimal places as desired.

Example 2: Find the value of $\sqrt{79{,}328}$ correct to the nearest hundredth.

Solution: 79328.0000)281.65+, *Ans.*

```
              4
 48       | 3 93
          | 3 84
 561         | 9 28
             | 5 61
 5626        | 3 67 00
             | 3 37 56
 56325          | 29 44 00
                | 28 16 25
```

It is clear that to find the square root of a number of 4 or 5 places, or more, neither the approximation method nor the table is of very much help; for that matter, neither is the slide rule. Only an extensive table of logarithms would do. Hence the "algebraic" method is quite useful, even though it may be slightly annoying at first.

Exercise 39.

Find, to the nearest tenth, the square root of each of the following by an approximation method; then check your result by means of the table:

1. 18	4. 41	7. 55	10. 150
2. 30	5. 60	8. 74	11. 172
3. 12	6. 21	9. 129	12. 200

Find, to the nearest tenth, the square root of each of the following by using the algebraic rule; check your result, by referring to the table:

13. 86	15. 6.25	17. 567	19. 69.43
14. 54.8	16. 382	18. 2933	20. 426.5

Evaluation of Formulas Involving Square Root. It so happens that formulas in mensuration, shop work and science problems frequently involve the square root of a constant or of one of the variables. If we wish to evaluate such a formula, it is useful to be able to find the numerical value of a square root quickly and easily; the following problems will afford practice in such computation.

Exercise 40.

1. Find the value of d in the formula $d=\sqrt{2hr}$ when $r=4.2$ and $h=6.5$.
2. Find the value of D in the formula $D=\sqrt{\frac{1}{3}RS}$ when $R=28$ and $S=6\frac{1}{4}$.
3. Find the value of t in the formula $t=\frac{44}{7}\sqrt{\frac{l}{g}}$ when $l=44.1$ and $g=9.80$
4. Find the value of A when $S=8.6$ in the formula $A=\frac{S^2}{4}\sqrt{3}$.
5. If $s=650$ and $g=32$, find the value of t to the nearest tenth in the formula $t=\sqrt{\frac{2s}{g}}$
6. Find b to the nearest hundredth from the formula $b=\sqrt{c^2+a^2}$ when $c=10.4$ and $a=4.2$.
7. The volume of a frustum of a pyramid is given by $V=\frac{1}{3}h(B_1+B_2+\sqrt{B_1B_2})$. Find V if $B_1=34$, $B_2=22$, and $h=6$.
8. The maximum distance at which an object on the earth may be seen from a point above the earth's surface is given by the formula $D=1.22\sqrt{E}$, where D is the distance of the object in miles and E is the elevation of the observer in feet. Find D when $E=840$ ft.
9. The "effective" area of a smokestack is expressed by the formula $E=A-\frac{3}{5}\sqrt{A}$, where A is its measured area in square feet. Find E when $A=44$ sq. ft.
10. The amount of sag (d) in a rope or wire suspended between two points is given by

$$d=\sqrt{\frac{3l(L-l)}{8}},$$

where l=length of the rope in feet when taut, L=its actual length in feet, and d equals the maximum sag in inches. Find d when $L=50$ and $l=45$.

Solving Equations Containing Radicals. Formulas and equations frequently involve radicals, particularly the square root of a quantity, whether a constant or a radical; or they contain terms that are raised to the second power. The procedure in such cases is illustrated by the following.

Example 1: If $A=\frac{\pi d^2}{4}$, find d.

Solution: $4A=\pi d^2$

$$d^2=\frac{4A}{\pi}$$

$$d=\sqrt{\frac{4A}{\pi}}=2\sqrt{\frac{A}{\pi}},\ \textit{Ans.}$$

Example 2: Solve for V: $F=\frac{wV^2}{gr}$

Solution: $grF=wV^2$

$$V^2=\frac{grF}{w}$$

$$V=\sqrt{\frac{grF}{w}},\ \textit{Ans.}$$

Example 3: If $a^2=b^2+c^2-2cp$, find b.

Solution: $b^2=a^2-c^2+2cp$

$$b=\sqrt{a^2-c^2+2cp},\ \textit{Ans.}$$

Example 4: In the formula $t=\sqrt{\frac{2s}{g}}$, solve for s.

Solution: $t^2=\frac{2s}{g}$

$$gt^2=2s$$

$$s=\frac{gt^2}{2},\ \textit{Ans.}$$

Example 5: Solve for r: $p=\frac{a}{2\pi\sqrt{nr}}$

Solution: $p^2=\frac{a^2}{4\pi^2 nr}$

$$4\pi^2 nrp^2=a^2$$

$$r=\frac{a^2}{4n\pi^2p^2},\ \textit{Ans.}$$

Exercise 41.

1. The kinetic energy of a body in motion is given by the formula $E=\frac{1}{2}mv^2$, where m is its mass and v its velocity. Solve for v.
2. The velocity in feet per second of a freely falling body at t seconds after it began falling is found from the formula $v=\frac{1}{2}gt^2$, where $g=$ the acceleration of gravity (32 ft. per sec.). Solve for t.

3. If m is the maximum "visibility" in miles from an elevation above the earth of h feet, we have $m=\sqrt{\frac{3h}{2}}$. Solve for h.
4. If h is the altitude of an equilateral triangle with side s, then $h^2=\frac{3s^2}{4}$. Solve for h.
5. Solve the formula $A=\frac{s^2}{4}\sqrt{3}$ for s.
6. The volume of a circular cone is given by $V=\frac{1}{3}\pi R^2h$. Solve for R.
7. In dealing with a radio circuit, the formula $f=\frac{1}{2\pi\sqrt{LC}}$ is used. Solve for C.
8. In computing the strength of a certain beam, the formula $\frac{I}{C^2}=\frac{\pi ab}{6}$ is needed. Solve for C.
9. The formula $P=\frac{Wv^2}{550\times 2g}$ is used in the determining horsepower. Solve for v.
10. A formula of electrical engineering is $C=\frac{E}{\sqrt{R^2+(2\pi nL)^2}}$. Find R.
11. The area in sq. ft. of the cross-section of a smoke stack required to carry off the smoke is given by $A=\frac{.6P}{\sqrt{h}}$, where P=no. of lb. of coal burned per hr., and h=height of stack in ft. Solve for h.
12. In Ex. 11, find A if $h=75$ ft. and $P=550$ lb.

Solving a Simple Quadratic Equation. A simple quadratic formula or equation is one which contains second-degree (squared) terms as well as first-degree terms, or only squared terms, involving one of the variables only, and no higher powers of any other variable; examples of such quadratic equations are:

(1) $y=3x^2$	(4) $H=I^2R$
(2) $y=5x^2+3x-10$	(5) $V=at+\frac{1}{2}at^2$
(3) $E=\frac{1}{2}mv^2$	(6) $V=\frac{1}{3}\pi R^2h$

The general type of a simple quadratic function may be written as

$$y=ax^2+bx+c,$$

and the general form of a simple quadratic equation is

$$ax^2+bx+c=o,$$

where a, b and c are the numerical coefficients. Thus, if the equation to be solved is $3x^2+5x-7=0$, then $a=3$, $b=5$, and $c=-7$.

The solution of such an equation is found by using the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Without explaining how this formula is derived, we shall illustrate its use in solving quadratic equations.

EXAMPLE 1: Solve for x: $3x^2+5x-7=0$.

SOLUTION:
$$x=\frac{-5\pm\sqrt{25-(4)(3)(-7)}}{(2)(3)},$$

$$\text{or } x=\frac{-5\pm\sqrt{25+84}}{6},$$

$$x=\frac{-5\pm\sqrt{109}}{6}=\frac{-5\pm 10.44}{6}.$$

$$\left.\begin{array}{l}\text{Thus } x\dfrac{-5+10.44}{6}=\dfrac{5.44}{6}=+.91;\\ \text{and also } x\dfrac{-5-10.44}{6}=\dfrac{-15.44}{6}=-2.57\end{array}\right\} Ans.$$

It should be noted from the above that there are *two values* for x; each of them satisfies the original equation. Every quadratic equation thus has *two roots,* although in the case of formulas, one of them might have no *practical* significance in the relation represented by the formula.

EXAMPLE 2: Solve to the nearest hundredth for t:

$$5t^2-10t+2=0$$

SOLUTION: In this case $a=5$, $b=-10$, and $c=2$.

$$\text{Then } t=\frac{+10\pm\sqrt{100-(4)(5)(2)}}{(2)(5)}$$

$$t=\frac{10\pm\sqrt{20}}{10}=\frac{10\pm 4.472}{10}$$

$t=+1.45$ and $+.55$, *Ans.*

NOTE: Before substituting in the formula, always transpose *all* terms to the left side of the equation; otherwise the signs of a, b and c will be incorrect.

Exercise 42.

Solve the following, finding the roots correct to the nearest tenth:

1. $3x^2-4x-11=0$
2. $x^2-30=10x$
3. $y^2+y=14$
4. $4k^2-k=8$
5. $2p^2-2=7p$
6. $32t+16t^2=240$

SQUARE ROOTS OF NUMBERS*

N	0	1	2	3	4	5	6	7	8	9
1.0	1.000	1.005	1.010	1.015	1.020	1.025	1.030	1.034	1.039	1.044
1	1.049	1.054	1.058	1.063	1.068	1.072	1.077	1.082	1.086	1.091
2	1.095	1.100	1.105	1.109	1.114	1.118	1.122	1.127	1.131	1.136
3	1.140	1.145	1.149	1.153	1.158	1.162	1.166	1.170	1.175	1.179
4	1.183	1.187	1.192	1.196	1.200	1.204	1.208	1.212	1.217	1.221
1.5	1.225	1.229	1.233	1.237	1.241	1.245	1.249	1.253	1.257	1.261
6	1.265	1.269	1.273	1.277	1.281	1.285	1.288	1.292	1.296	1.300
7	1.304	1.308	1.311	1.315	1.319	1.323	1.327	1.330	1.334	1.338
8	1.342	1.345	1.349	1.353	1.356	1.360	1.364	1.367	1.371	1.375
9	1.378	1.382	1.386	1.389	1.393	1.396	1.400	1.404	1.407	1.411
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446
1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480
2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513
3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546
4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609
6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640
7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670
8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700
9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729
3.0	1.732	1.735	1.738	1.741	1.744	1.746	1.749	1.752	1.755	1.758
1	1.761	1.764	1.766	1.769	1.772	1.775	1.778	1.780	1.783	1.786
2	1.789	1.792	1.794	1.797	1.800	1.803	1.806	1.808	1.811	1.814
3	1.817	1.819	1.822	1.825	1.828	1.830	1.833	1.836	1.838	1.841
4	1.844	1.847	1.849	1.852	1.855	1.857	1.860	1.863	1.865	1.868
3.5	1.871	1.873	1.876	1.879	1.881	1.884	1.887	1.889	1.892	1.895
6	1.897	1.900	1.903	1.905	1.908	1.910	1.913	1.916	1.918	1.921
7	1.924	1.926	1.929	1.931	1.934	1.936	1.939	1.942	1.944	1.947
8	1.949	1.952	1.954	1.957	1.960	1.962	1.965	1.967	1.970	1.972
9	1.975	1.977	1.980	1.982	1.985	1.987	1.990	1.992	1.995	1.997
4.0	2.000	2.002	2.005	2.007	2.010	2.012	2.015	2.017	2.020	2.022
1	2.025	2.027	2.030	2.032	2.035	2.037	2.040	2.042	2.045	2.047
2	2.049	2.052	2.054	2.057	2.059	2.062	2.064	2.066	2.069	2.071
3	2.074	2.076	2.078	2.081	2.083	2.086	2.088	2.090	2.093	2.095
4	2.098	2.100	2.102	2.105	2.107	2.110	2.112	2.114	2.117	2.119

*From Mechanical Engineers' Handbook, by Lionel S. Marks (1941). Courtesy of the McGraw-Hill Book Co.

N	0	1	2	3	4	5	6	7	8	9
4.5	2.121	2.124	2.126	2.128	2.131	2.133	2.135	2.138	2.140	2.142
6	2.145	2.147	2.149	2.152	2.154	2.156	2.159	2.161	2.163	2.166
7	2.168	2.170	2.173	2.175	2.177	2.179	2.182	2.184	2.186	2.189
8	2.191	2.193	2.195	2.198	2.200	2.202	2.205	2.207	2.209	2.211
9	2.214	2.216	2.218	2.220	2.223	2.225	2.227	2.229	2.232	2.234
5.0	2.236	2.238	2.241	2.243	2.245	2.247	2.249	2.252	2.254	2.256
1	2.258	2.261	2.263	2.265	2.267	2.269	2.272	2.274	2.276	2.278
2	2.280	2.283	2.285	2.287	2.289	2.291	2.293	2.296	2.298	2.300
3	2.302	2.304	2.307	2.309	2.311	2.313	2.315	2.317	2.319	2.322
4	2.324	2.326	2.328	2.330	2.332	2.335	2.337	2.339	2.341	2.343
5.5	2.345	2.347	2.349	2.352	2.354	2.356	2.358	2.360	2.362	2.364
6	2.366	2.369	2.371	2.373	2.375	2.377	2.379	2.381	2.383	2.385
7	2.387	2.390	2.392	2.394	2.396	2.398	2.400	2.402	2.404	2.406
8	2.408	2.410	2.412	2.415	2.417	2.419	2.421	2.423	2.425	2.427
9	2.429	2.431	2.433	2.435	2.437	2.439	2.441	2.443	2.445	2.447
6.0	2.449	2.452	2.454	2.456	2.458	2.460	2.462	2.464	2.466	2.468
1	2.470	2.472	2.474	2.476	2.478	2.480	2.482	2.484	2.486	2.488
2	2.490	2.492	2.494	2.496	2.498	2.500	2.502	2.504	2.506	2.508
3	2.510	2.512	2.514	2.516	2.518	2.520	2.522	2.524	2.526	2.528
4	2.530	2.532	2.534	2.536	2.538	2.540	2.542	2.544	2.546	2.548
6.5	2.550	2.551	2.553	2.555	2.557	2.559	2.561	2.563	2.565	2.567
6	2.569	2.571	2.573	2.575	2.577	2.579	2.581	2.583	2.585	2.587
7	2.588	2.590	2.592	2.594	2.596	2.598	2.600	2.602	2.604	2.606
8	2.608	2.610	2.612	2.613	2.615	2.617	2.619	2.621	2.623	2.625
9	2.627	2.629	2.631	2.632	2.634	2.636	2.638	2.640	2.642	2.644
7.0	2.646	2.648	2.650	2.651	2.653	2.655	2.657	2.659	2.661	2.663
1	2.665	2.666	2.668	2.670	2.672	2.674	2.676	2.678	2.680	2.681
2	2.683	2.685	2.687	2.689	2.691	2.693	2.694	2.696	2.698	2.700
3	2.702	2.704	2.706	2.707	2.709	2.711	2.713	2.715	2.717	2.718
4	2.720	2.722	2.724	2.726	2.728	2.729	2.731	2.733	2.735	2.737
7.5	2.739	2.740	2.742	2.744	2.746	2.748	2.750	2.751	2.753	2.755
6	2.757	2.759	2.760	2.762	2.764	2.766	2.768	2.769	2.771	2.773
7	2.775	2.777	2.778	2.780	2.782	2.784	2.786	2.787	2.789	2.791
8	2.793	2.795	2.796	2.798	2.800	2.802	2.804	2.805	2.807	2.809
9	2.811	2.812	2.814	2.816	2.818	2.820	2.821	2.823	2.825	2.827
8.0	2.828	2.830	2.832	2.834	2.835	2.837	2.839	2.841	2.843	2.844
1	2.846	2.848	2.850	2.851	2.853	2.855	2.857	2.858	2.860	2.862
2	2.864	2.865	2.867	2.869	2.871	2.872	2.874	2.876	2.877	2.879
3	2.881	2.883	2.884	2.886	2.888	2.890	2.891	2.893	2.895	2.897
4	2.898	2.900	2.902	2.903	2.905	2.907	2.909	2.910	2.912	2.914

N	0	1	2	3	4	5	6	7	8	9
8.5	2.915	2.917	2.919	2.921	2.922	2.924	2.926	2.927	2.929	2.931
6	2.933	2.934	2.936	2.938	2.939	2.941	2.943	2.944	2.946	2.948
7	2.950	2.951	2.953	2.955	2.956	2.958	2.960	2.961	2.963	2.965
8	2.966	2.968	2.970	2.972	2.973	2.975	2.977	2.978	2.980	2.982
9	2.983	2.985	2.987	2.988	2.990	2.992	2.993	2.995	2.997	2.998
9.0	3.000	3.002	3.003	3.005	3.007	3.008	3.010	3.012	3.013	3.015
1	3.017	3.018	3.020	3.022	3.023	3.025	3.027	3.028	3.030	3.032
2	3.033	3.035	3.036	3.038	3.040	3.041	3.043	3.045	3.046	3.048
3	3.050	3.051	3.053	3.055	3.056	3.058	3.059	3.061	3.063	3.064
4	3.066	3.068	3.069	3.071	3.072	3.074	3.076	3.077	3.079	3.081
9.5	3.082	3.084	3.085	3.087	3.089	3.090	3.092	3.094	3.095	3.097
6	3.098	3.100	3.102	3.103	3.105	3.106	3.108	3.110	3.111	3.113
7	3.114	3.116	3.118	3.119	3.121	3.122	3.124	3.126	3.127	3.129
8	3.130	3.132	3.134	3.135	3.137	3.138	3.140	3.142	3.143	3.145
9	3.146	3.148	3.150	3.151	3.153	3.154	3.156	3.158	3.159	3.161
10	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302
1	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450
2	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592
3	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728
4	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860
15	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.987
6	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111
7	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231
8	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347
9	4.359	4.370	4.382	4.393	4.405	4.416	4.427	4.438	4.450	4.461
20	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572
1	4.583	4.593	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680
2	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.764	4.775	4.785
3	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889
4	4.899	4.909	4.919	4.930	4.940	4.950	4.960	4.970	4.980	4.990
25	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089
6	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187
7	5.196	5.206	5.215	5.225	5.235	5.244	5.254	5.263	5.273	5.282
8	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376
9	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468
30	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559
1	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648
2	5.657	5.666	5.675	5.683	5.692	5.701	5.710	5.718	5.727	5.736
3	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822
4	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.908

SQUARE ROOTS (*continued*)

N	0	1	2	3	4	5	6	7	8	9
35	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992
6	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	0.066	6.075
7	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156
8	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237
9	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317
40	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395
1	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473
2	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550
3	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626
4	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701
45	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775
6	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848
7	6.856	6.863	6.870	6.877	6.885	6.892	6.899	6.907	6.914	6.921
8	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993
9	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064
50	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134
1	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204
2	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273
3	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342
4	7.348	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.409
55	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477
6	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543
7	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609
8	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675
9	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740
60	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804
1	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868
2	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931
3	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994
4	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056
65	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118
6	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179
7	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240
8	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301
9	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361
70	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420
1	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479
2	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538
3	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597
4	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654

SQUARE ROOTS (*continued*)

N	0	1	2	3	4	5	6	7	8	9
75	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712
6	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769
7	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826
8	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883
9	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939
80	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994
1	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050
2	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105
3	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160
4	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214
85	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268
6	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322
7	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375
8	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429
9	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482
90	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534
1	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586
2	9.592	9.597	9.602	9.607	9.612	9.618	9.623	9.628	9.633	9.638
3	9.644	9.649	9.654	9.659	9.664	9.670	9.675	9.680	9.685	9.690
4	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742
95	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793
6	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844
7	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894
8	9.899	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945
9	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995

10. VARIATION, DEPENDENCE AND GRAPHS

Another Use for the Formula. By studying a formula carefully it is possible to tell how changes in the value of one of the quantities will affect the values of another quantity. Thus we note that in the formula

$$D=\frac{M}{V},$$

(1) if V remains constant and M is doubled, then D is doubled.
(2) if M remains constant and V is doubled, D becomes ½ as large.
(3) if M and V are both doubled, then D remains the same.
(4) if M is doubled and V is halved, then D becomes 4 times as large.

Or again, consider the formula $A=6e^2$;

(1) if e is doubled, A becomes 4 times as large.
(2) if e is halved, A becomes ¼ as large.
(3) if e is multiplied by 3, A becomes 9 times as large.
(4) if e is divided by 4, A becomes $\frac{1}{16}$ as large.
(5) if e is multiplied by 10, A becomes 100 times as large.

Exercise 43.

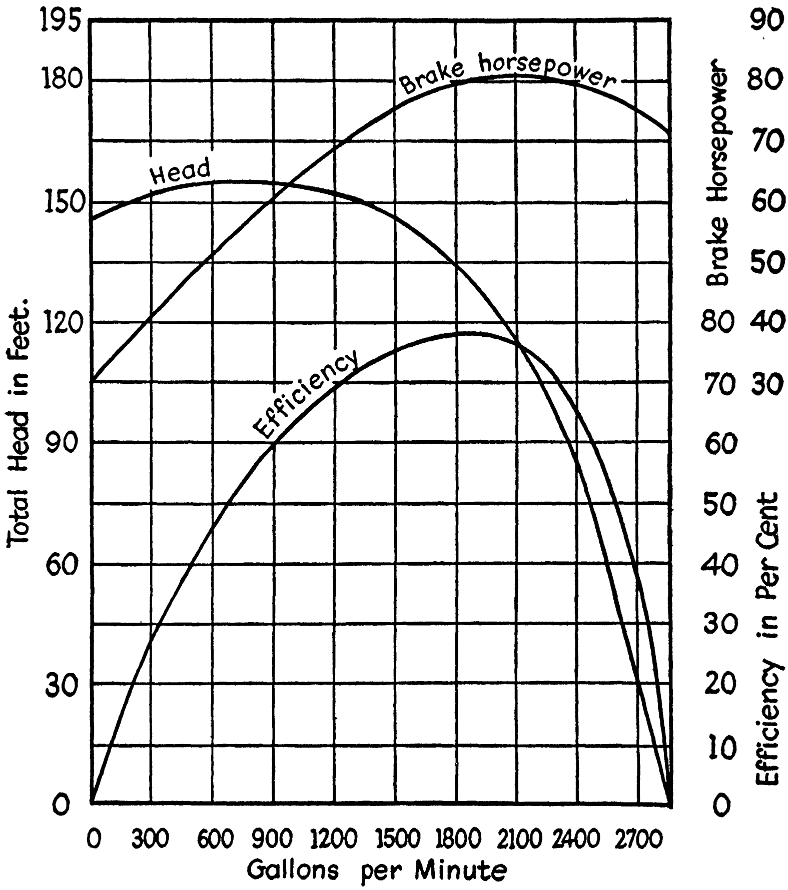
1. In the formula $D=RT$, how is D changed when
   (a) R is doubled and T remains constant.
   (b) T is halved and R remains constant.
   (c) R is divided by 10 and T remains constant.
   (d) T is tripled and R is halved.
   (e) R is divided by 4 and T is divided by 2.

2. In the formula $A=\frac{1}{2}bh$, how is A affected when
   (a) b and h are both tripled.
   (b) h remains constant and b is divided by 6.
   (c) b is doubled and h is halved.
   (d) h is multiplied by 4 and b remains constant.

3. In the formula $PV=k$, if k is always constant, what change takes place when
   (a) P increases.
   (b) V increases.
   (c) V is halved.
   (d) P is tripled.
   (e) V is multiplied by 5.

4. In the formula $A=\pi R^2$, what happens to A when
   (a) R is doubled.
   (b) R is divided by 3.
   (c) R is multiplied by 5.
   (d) R is divided by 10.

What a Formula Does. By this time it will be seen that a formula has a number of advantages over the verbal statement of the relationship between two or more quantities. It is briefer and simpler; it emphasizes the nature of the relationship; it enables us to compute the value of one of the variables when specific values of the other variables are known; it permits the relationship to be restated so that any particular variable may be expressed or "described" in terms of the other variables, which is generally a convenient device. In short, a formula is a powerful mathematical tool, since it implicitly represents *all the infinite sets of corresponding values of the variables,* all at once, so to speak.



Characteristic curves of a single-stage centrifugal pump operating at a constant speed of 1,200 r.p.m.

A Graph also Shows Dependence. Consider the three graphs shown in the accompanying figure. Study the curve labeled "efficiency"; referring to the horizontal scale of "gallons per minute" and the vertical scale (at the lower right) of "efficiency in per cent," we can read directly from the efficiency at any particular load, or the load at any particular efficiency. Thus when pumping 2700 gal. per minute the efficiency is 40%; when the efficiency is 60%, it may be pumping a little over 900, or about 2500 gal. per minute (the 60% line crosses the curve in *two places*). Also the maximum (greatest) efficiency is reached when pumping a little under 1950 gal. per minute. And in the same way, the other two curves might be studied and analyzed. This could not be done conveniently with a formula.

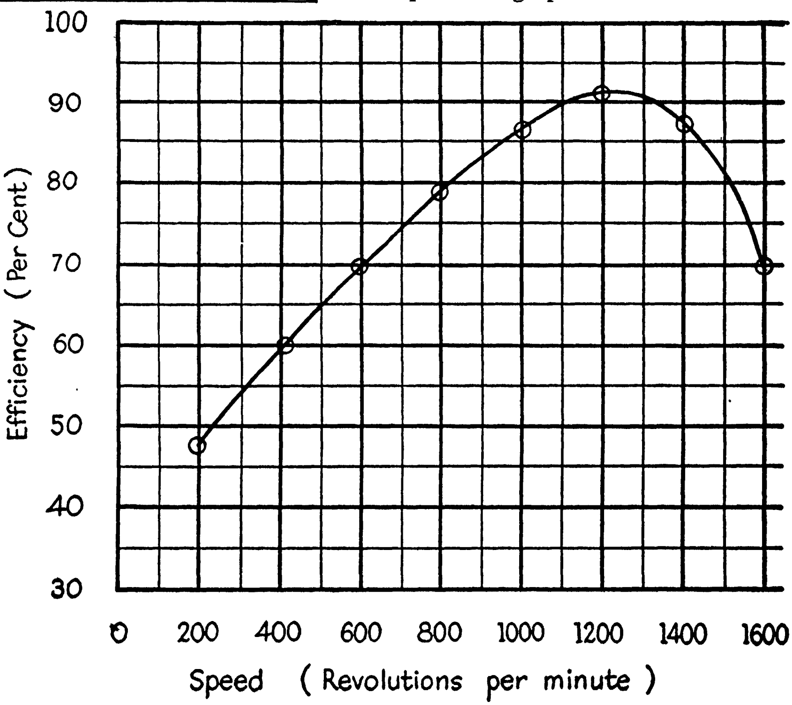
What a Graph Can Do. A graph, as the name itself suggests, can go a step further than the formula—it can make visible what the formula represents—it can give an actual *picture* of the mathematical relationship. The relationship literally becomes more graphic; the relative magnitudes of the variables become apparent to the eye, as do extreme maximum and minimum values, if any; so do the rates at which they change; trends become clear; extrapolation and interpolation become more meaningful; any special features of the relationship are emphasized; general types of relationships are recognizable; two or more relationships can frequently be directly compared with one another.

Mathematical Functions. In mathematics, a definite quantitative relation between two or more variables, whether expressed verbally, by a formula, or by a graph, is called a *functional relationship,* or simply a mathematical function. Each variable is said to be a function of the other. The word function, as used here, has nothing to do with use or purpose; it simply calls attention to the fact that the quantities in question are quantitatively related to each other in a definite manner. Each, in other words, depends upon the other (or others) for its numerical values. Thus if $V=32t^2$, then V is a function of t, and t is a function of V; or, the value of V depends upon the particular value taken for t; and the value of t depends upon the value of V.

It should be noted that in mathematics the word *curve* is used to designate any sort of a graph between two related variables, even though that graph may be a straight line, and not really "curved" at all. It also frequently happens that two variables, representing physical quantities or scientific measurements, may indeed be mutually dependent one upon the other, i.e., functionally related, and yet there is no simple or known formula to represent the relation between them. Such a relationship is called an *empirical function,* and its graph is called an empirical curve. Before discussing formula graphs (i.e., mathematical functions), we shall study empirical graphs a little further in order to understand mathematical graphs better.

Empirical Curves. Consider the accompanying table of values, which were determined by experimental measurements. It represents certain particular pairs of corresponding values of the two variables in question, which in this case are the speed of a certain machine (expressed in revolutions per minute) and the efficiency of the machine (expressed in per cents). Using the horizontal scale or axis for values of the speed (reading from left to right) and using the vertical axis for the efficiencies (reading from the bottom up), we then plot the graph as here shown.

Speed	Efficiency (%)
200	47.5
400	60.0
600	70.0
800	79.0
1000	86.0
1200	90.5
1400	87.5
1600	70.0



Exercise 44.

1. The solubility (S) of a certain chemical, as shown by the number of grams dissolved in a given amount of water, varied with the temperature (T) as shown in Table I. Plot the graph, using the horizontal scale for the values of the S. How many grams dissolved at 55°? At what temperature would 27 gm. be dissolved?

Table I

T	S
10	10 gm.
20	12
30	16
40	21
50	30
60	50
65	65

Table II

T	W
5	6.8
10	9.4
15	12.8
20	17.2
25	22.9
30	29.2

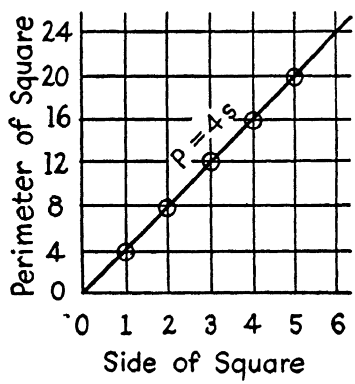
Table III

H	P
0	30.0
5,000	24.8
10,000	21.2
15,000	17.0
20,000	14.5
25,000	11.4
30,000	9.5
35,000	7.8

2. The weight (W gm.) of water contained in a cubic meter of saturated air at various temperatures (T) is given in Table II. Plot the graph with the temperature along the horizontal scale. How many grams are contained in a cubic meter of air at 18°? at 28°? At what temperature does one cubic meter of saturated air contain 20 gms.? 25 gms.?
3. The atmospheric pressure (P) expressed in inches of mercury varies at different altitudes above sea level as shown in Table III. Plot the graph, using the vertical scale for the pressure. What is the pressure at the top of Mount Everest, 29,000 ft. high? How high is an airplane when its pressure gage indicates 22.5 inches?

The Linear Function. When two variables are so related that their graph is a straight line, the relation is said to be a *linear function.* Consider, for example, the relation between the side of a square and its perimeter. Whatever the length of its side, its perimeter is always 4 times as great; or, $P=4s$.

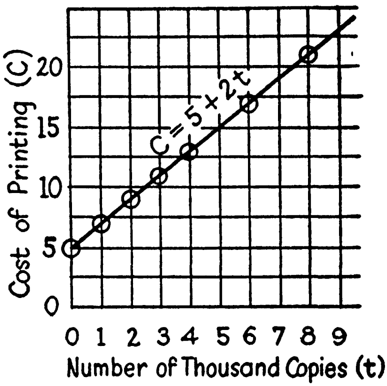
s	P
0	0
1	4
2	8
2½	10
3	12
4	16
5	20



When $s=0$, $P=0$; since, if there is no square, there can be no perimeter. Plotting these values on ruled paper, we obtain the accompanying linear graph. From this graph it is possible to "read off" many pairs of values of s and P that do not appear in the original table.

Or again, consider the cost of printing circulars, quoted at \$5 for setting the type, and \$2 per 1000 copies run off. The cost can be represented by the equation $C=5+2t$, where t=the number of thousand copies printed, and C=the total cost. The following set of values can readily be tabulated from the formula. When the graph of this table is

t	C
0	5
1	7
2	9
3	11
4	13
6	17
8	21



plotted, the curve is again seen to be a straight line. This time, however, it does not pass through the "0,0 point," but at the point where $t=0, C=5$. This, of course, mathematically represents the fact that even if no copies are printed the cost of setting type is \$5.

Exercise 45.

1. Plot the graph of $I=2.5C$, representing the relationship between inches (I) and centimeters. Use values of $C=0$, 2, 4, 6, 10. Use the horizontal axis for values of C.
2. Draw the graph of $E=IR$, considering a constant value of $R=20$; i.e., plot the graph of $E=20I$. Use values of $I=0$, 1, 2, 3, 4, 5, plotting them along the horizontal axis.
3. Plot the graph of $v=gt$, where v is the velocity of a freely falling body in ft. per second, and t is the time in seconds. Consider g as constant and equal to 32 ft. per sec. Use values of $t=0$, 1, 2, 3, 5, 8, 10; use the horizontal axis for t.
4. Draw the graph of $D=RT$, where R is constant and equal to 60 miles per hr., T is the time in hours, and D is the distance covered. Plot T along the horizontal axis.
5. The circumference (C) of a circle is related to its diameter (D) as given by the formula $C=\pi D$, where $\pi=22/7$. Plot the graph, using values of $D=0$, 7, 14, 21, 28, 35, along the horizontal axis.
6. The amount due at simple interest on \$100 at 3% a year for various periods of time is given by the formula $A=100+3n$, where n is the number of years. Plot the graph from $n=0$ to $n=6$; use the horizontal axis for n.

7. Plot the graph of $F=\frac{9}{5}C+32$, where C=the temperature expressed in Centrigrade degrees, and F=the temperature in Fahrenheit degrees. Use the horizontal axis for values of C.
8. Plot the graph of $V=v_0+at$, where V is the final velocity in ft. per sec. and t is the time in seconds, and where v_0, the initial velocity, is 600, and a, the acceleration, is 50 ft. per sec. Use values of $t=0, 2, 4, 6, 8, 10$ along the horizontal axis.

Direct Variation. Dependence or variation of this type, where the graph of the relation is a straight line, is called *direct variation.* The significance of the word "direct" is that as one variable increases, the other also increases in direct proportion, just as we saw in the section on ratio and proportion (Chap. I, Sect. 6). Whenever two variables *vary directly,* the relation may be expressed as

$$y=kx, \text{ or } \frac{y}{x}=k;$$

in this latter form it is seen that the *ratio* of the variables has a constant value, whatever their *individual* values may be. Recalling that the volume and temperature of a gas (at constant pressure) were found to be in *direct proportion;* thus

$$\frac{V_1}{T_1}=\frac{V_2}{T_2}, \text{ which can also be written as}$$

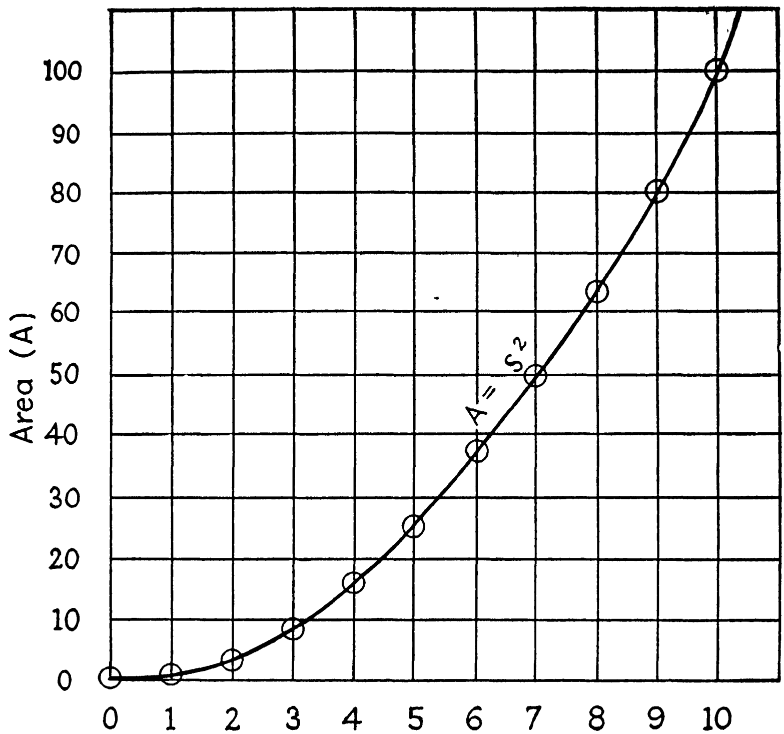
$$\frac{V_1}{T_1}=\frac{V_2}{T_2}=\frac{V_3}{T_3}=\frac{V_4}{T_4}= \ldots\ldots = k;$$

this means that the ratio of *any* volume to its *corresponding* temperature is the same ratio as *any other* volume is to *its* corresponding temperature; or, the *ratio being constant,* the relation is seen as an example of direct variation if it is simply rewritten as

$$\frac{V}{T}=k, \text{ or } V=kT.$$

The Parabolic Function. Some variables are related by a mathematical dependence which involves the square of one of them, as e.g., the relation between the area of a square and the length of its side, or $A=s^2$. Thus, from this formula we get the following table of values; then, from this tables of values we can plot the graph shown below. It is called a *parabolic* curve.

s	0	1	2	3	4	5	8	10
A	0	1	4	9	16	25	64	100

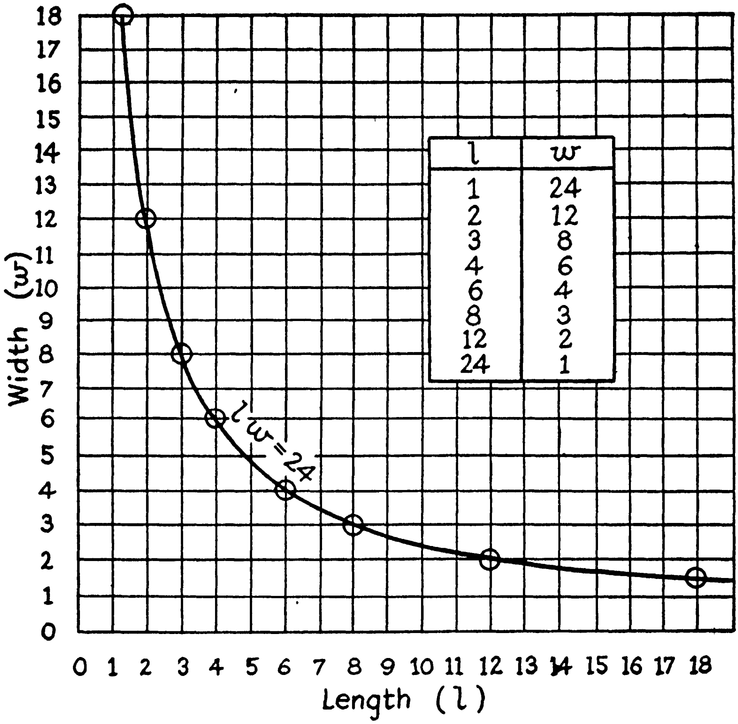


Exercise 46.

Plot the graph of each of the following formulas; in each case use the horizontal axis for the variable that is squared.

1. $S=at^2$, where S is the distance (S) in feet moved by a body in t seconds at a constant acceleration $a=10$ ft. per sec. Use values of $t=0, 1, 2, 3, 4$.
2. $A=\pi R^2$, where A is the area of a circle, R=radius, and $\pi=22/7$. Use values of $R=0, 3\frac{1}{2}, 7, 10\frac{1}{2}, 14$.
3. $H=I^2R$, where H is the heat produced by an electric current (I) that varies; consider the R constant and equal to 10. Use values of $I=0, 1, 2, 3, 4, 5$.
4. The kinetic energy (E) of a moving automobile weighing 2,000 lb. is $E=\frac{1}{2}mv^2$, where $m=$ the weight of the car and v its velocity in ft. per sec. Plot the graph for values of $v=0, 20, 40, 60, 80$.
5. The wind pressure (P) on a moving plane at 90° inclination, expressed in lb. per sq. ft., is given by the formula $P=.003\ v^2$, where v is the wind velocity expressed in miles per hour. Plot the graph for values of $v=0, 10, 20, 30, 40, 50$.

The Hyperbolic Function. Another type of function frequently encountered is the *hyperbolic* relation, expressing inverse variation, such as the



l	w
1	24
2	12
3	8
4	6
6	4
8	3
12	2
24	1

relations $xy=k$, $A=lw$, or $R=\frac{D}{T}$. In each case, if one of the variables is held constant, the graph takes the characteristic form shown above. Consider first, the formula $A=lw$, where l and w are the length and width, respectively, of a rectangle having the area A. If we assume that the area A is constant, equal say to 24 sq. in., then the graph of the formula lw 24 is found as follows, since $l=\frac{24}{w}$, or $w=\frac{24}{l}$. This is the type of problem encountered by photoengravers in determining the possible dimensions for a rectangle of a given, constant area.

Inverse Variation. In this type of relation, neither variable can ever equal zero, although it can become as small as you please. The smaller either variable becomes, the larger is the value of the other, since their product is a constant; that is what is meant by *inverse variation.* It is exactly the same situation as was seen in the case of inverse proportion (Chap. I, Sect. 6), where the pressure and volume of a gas at constant temperature were in *inverse* proportion, $\frac{P_1}{P_2}=\frac{V_2}{V_1}$ This can also be written as follows:

$$P_1 V_1=P_2 V_2=P_3 V_3=P_4 V_4= \ldots =k;$$

i.e., *any* P multiplied by its *corresponding* V gives the *same constant product* as *any other* P multiplied by *its* corresponding V. Hence the formula is $PV=k$, which is, as we now know, a hyperbolic curve.

Exercise 47.

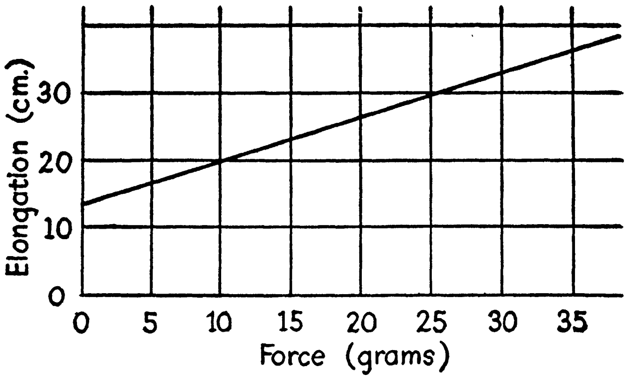
Plot the graph of each of the following formulas:

1. $xy=20$
2. $y=\frac{144}{x}$
3. $IR=120$
4. $PV=1000$
5. $T=\frac{100}{R}$
6. $500=Fs$

Related Variables. The dependence of varying quantities upon one another has already been discussed in connection with the formula. The value of either of two variables may properly be said to "depend" upon the other for its value. Quantities depending upon each other in this way are said to bear a functional relationship to each other; each is said to be a function of the other. Thus the volume occupied by a gas at constant temperature depends upon the pressure exerted; conversely, the pressure exerted depends upon the volume occupied by the gas. Hence V is a function of P, or P is a function of V. Similarly, the amount of expansion due to heat depends upon the temperature; the population of a community, upon the time; the fuel consumed in driving a locomotive, upon the speed; the quantity of a commodity sold, upon the price. In each instance the latter variable is conveniently regarded as the independent, and the former as the dependent, variable.

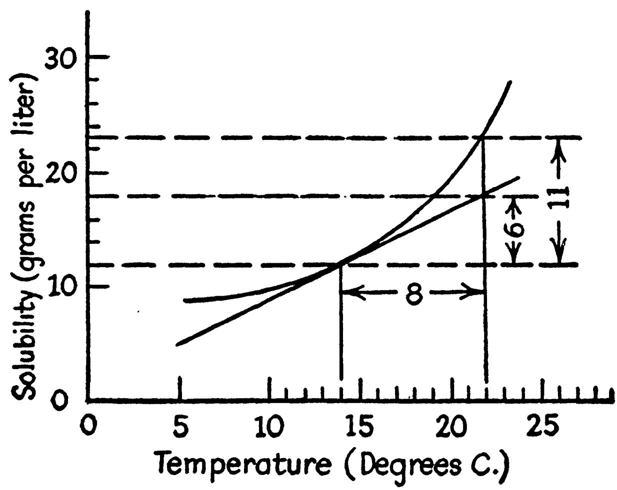
According to accepted convention, the independent variable is denoted by x and its values are always measured along the horizontal scale, while the dependent variable is denoted by y, and its values are measured along the vertical scale. Hence, the independent variable may be regarded as freely assuming all values along the range of the horizontal axis, while the dependent variable necessarily varies in some definite way to correspond, as shown by the *changing height* of the graph.

The Rate of Change Concept. It is clear that for any given point on a graph, its horizontal distance from the vertical scale (abscissa) represents the magnitude of the independent variable, while the vertical distance above or below the horizontal scale (ordinate) represents the corresponding magnitude of the dependent variable. Thus the position of the curve with respect to the axes depicts the actual magnitudes of the variables. But in studying changing variables and functional relationships, it is frequently desirable to inquire as to the *rate* at which a quantity



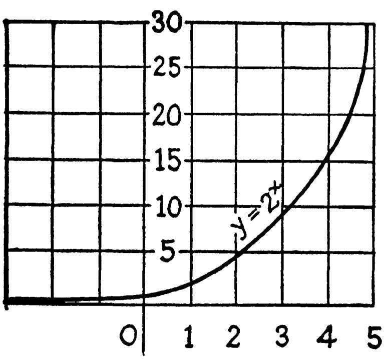
is changing, i.e., how fast it is increasing or decreasing, rather than how large or how small it is. Rate implies a ratio; a rate of change means the amount of change in the function (or dependent variable) per unit change in the independent variable. On the graph this means the amount of vertical rise or fall in the curve per horizontal unit, so that when discussing rates of change we are concerned with the *steepness* of the graph rather than with its actual height at any particular point. If the graph of a given function is a straight line, the function is obviously increasing at a constant rate, since the increase (or decrease) in the dependent variable is the same for every horizontal unit. Most quantities, however, change at varying instead of at constant rates, so that it becomes necessary to distinguish between an *average* rate and an *instantaneous* rate.

Average Rate of Change. An average rate during a definite interval does not necessarily imply that the rate during that interval need be uniform;



indeed, it may fluctuate considerably. An average rate during an interval merely means *the ratio of the net increase (or decrease) of the dependent variable during a given interval to the net change of the independent variable.* To find an average rate graphically, it is only necessary to read the amount of increase (or decrease) at the end of a given convenient interval and divide by the length of that interval. For example, the amount by which the solubility of a certain substance changed during the interval from 14° to 22° is seen to be 11 grams; hence the average rate is $11/8$ or 1.38 grams per degree. It is clear that the average rate of change will depend upon the length of the interval during which the average rate is observed, as well as where the interval is chosen.

The Exponential Function. One of the most interesting and important of all functions is the *exponential function,* $y=k^x$. It will be observed that it differs markedly from the ordinary *power function,* $y=x^k$; in the latter the exponent is merely a constant, while in the former it is the independent variable itself. Let us study carefully the graph of $y=2^x$. At first the curve rises very slowly, but gradually its rate of increase becomes greater and greater; as a matter of fact, the greater the value of x, the more rapid is the rate of growth of y. In this respect it resembles the graph of $A=P(1+i)^n$, which is the formula for the amount at compound interest, and is also of the type $y=k^x$. In both these curves, the *rate of increase of the dependent variable at any instant is proportional to the magnitude of the independent variable at the particular instant.* This is an extremely important characteristic of every exponential function. Otherwise expressed, the percentage of increase is constant throughout. Consequently any function of the type $y=ak^x$ is said to follow the *compound interest* law, or more picturesquely, the "snowball law," since it resembles the growth of a snowball rolling down a hill; it gathers more and more snow the farther down it rolls, and at any given instant is growing at a rate which is approximately proportional to the magnitude which it has already attained at that instant.



Graph of $y = 2^x$

Exponential functions arise in connection with the speed of some chemical reactions, the rate of decomposition of radium, the reduction in speed of revolving wheels, the amount of belt friction on drums, the flow of electric currents, the transmission of light under certain conditions, changes in atmospheric pressure with increase in elevation, and many other physical and technical phenomena.

11. LOGARITHMS

Logarithms Are Exponents. Consider the exponential equation $N=10^x$. How large is N? That depends upon the particular value selected for x; thus

$$\begin{aligned} &\text{if } x=1, && N=10 \\ &\text{if } x=2, && N=100 \\ &\text{if } x=3, && N=1000 \\ &\text{if } x=4, && N=10{,}000 \end{aligned}$$

etc.

What would be the value of N if $x=1.5$? Clearly, since 1.5 is greater than 1, N would be more than 10; and since 1.5 is less than 2, N would be less than 100. In other words, when $x=1.5$, the value of N must be *between* 10 and 100. For any value of x equal to 1 plus some decimal, the value of N must be between 10 and 100. Or again, if we know that the value of N is some number between 100 and 1000, we know at once that x must have a value somewhere between 2 and 3.

In the exponential form of the equation $N=10^x$, we refer to N as the *number;* to 10, as the *base;* and to the exponent (x) as the *logarithm of the number N to the base 10.* This may be written as $x=\log_{10}N$. When expressed in this way, we speak of it as being in the "logarithmic form"; when written as $N=10^x$ we speak of it as the "exponential form." But the two equations are equivalent and interchangeable—two different ways of writing the same relationship.

Any number could be used as a base; thus

$$y=2^x;\ \text{or}\ \log_2 y=x$$
$$P=5^y;\ \text{or}\ \log_5 P=y$$
$$A=k^t;\ \text{or}\ \log_k A=t$$

For practical computational purposes, however, the base 10 is universally used, and for this reason the base is commonly omitted. Thus we write

$$\log 10{,}000=4, \text{ instead of } \log_{10}10{,}000=4;$$
$$\text{or}\quad \log P=x, \text{ instead of } \log_{10}P=x.$$

In other words, the logarithm of a given number is the exponent to which 10 must be raised to give that number. Logarithms are thus usually not whole numbers, but mixed numbers, and are invariably expressed in decimal form. Every number has a logarithm; also, every logarithm corresponds to some number.

Characteristic and Mantissa. From what has been said thus far, it is clear that very few logarithms are whole numbers—by far most of them are decimals whose approximate values are to be found in a table.

The part of the logarithm to the *left* of the decimal point, i.e., the integral part, is known as the *characteristic;* the decimal part is called the *mantissa.* Thus, the value of log 634=2.8021; the characteristic is "2" and the mantissa is ".8021."

Rule 1: If the number whose logarithm is being found is greater than 1, the characteristic is positive, and numerically *one less than the number of figures to the left of the decimal point* in the given number.

For example:

log 7.92 = .8987
log 79.2 =1.8987
log 792 =2.8987
log 79,200 =4.8987

Rule 2: If the number whose logarithm is to be found is less than 1, the characteristic is then negative and numerically *one greater than the number of zeros immediately following the decimal point;* but the mantissa is still positive. The two parts of the logarithm are not actually added together, but are written as shown below.

For example:

log .792 =—1.0+.8987=9.8987—10
log .0792 =—2.0+.8987=8.8987—10
log .00792=—3.0+.8987=7.8987—10

Finding a Logarithm from the Table. From these rules and illustrations it will be seen that the characteristic of a logarithm is determined solely by the position of the decimal point in the given number, and not by the particular sequence of digits; the mantissa, however, is quite independent of the position of the decimal point, depending instead upon the actual sequence of digits in the given number. Thus, for example, the mantissas of the respective logarithms of 264,500, 26.45, and .0002645 are identical; their logarithms differ only in the characteristics, and are, respectively, 5.4224, 1.4224, and 6.4224—10. In the "table of logarithms" only the mantissas are shown, the decimal point in front of each figure in the table being understood; the characteristic is supplied mentally, by inspection.

EXAMPLE 1: Find from the table the value of log 467.

SOLUTION: Look under the column headed "N," running down to "46"; then run across horizontally until reaching the column headed "7." The figure found is "6693." Hence log 467= 2.6693, since the 467 has 3 digits, and the required characteristic is 2.

EXAMPLE 2: Find log .0803.

SOLUTION: Look under "N" for 80; opposite 80, under "3," find "9047." Hence log .0803=8.9047—10.

TABLE OF LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2801	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
N	0	1	2	3	4	5	6	7	8	9

TABLE OF LOGARITHMS (*continued*)

N	0	1	2	3	4	5	6	7	8	9
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N.	0	1	2	3	4	5	6	7	8	9

Interpolation. If the number whose logarithm is sought should contain more than three significant figures, the logarithm is found by a method known as *interpolation.* This is based upon the idea of direct proportion; i.e., it is assumed that the logarithm of a number "halfway between" two numbers is halfway between their logarithms; etc. Although such direct variation between numbers and their logarithms is not strictly true, the approximation is accurate enough for ordinary purposes of calculation.

EXAMPLE 1: Find log 2834.

SOLUTION: log 2830=3.4518
log 2840=3.4533
Interval between numbers="10"
Difference between logarithms=.0015
Given number N=2834 is .4 of the interval 10;
.4×.0015=.0006.
Hence, log 2834=3.4518+.0006=3.4524, *Ans.*

EXAMPLE 2: Find log .06178.

SOLUTION: log .06170=8.7903—10
log .06180=8.7910—10
Difference between logs=.0007
Given N=.06178 is .8 of "interval" between .06170 and .06180; thus .8×.0007=.00056, or .0006, is added to the mantissa .7903, giving .7909.
Hence log .06178=8.7909—10, *Ans.*

EXAMPLE 3: Find log 12.476

SOLUTION: log 12.4=1.0934
log 12.5=1.0969
Difference between logs=.0035
.76×.0035=.0027.
Hence log 12.476=1.0961, *Ans.*

Exercise 48.

Find the logarithms of the following numbers:

1. 932	5. .000526	9. .6932
2. 408.5	6. 643,000	10. .00045
3. .504	7. 8.037	11. 4,806
4. 71.68	8. .00341	12. 236.96

Finding the Number When Its Logarithm Is Given. If the logarithm of a number is given and we wish to determine the number itself, this reverse process is called finding the *antilogarithm.* The principle of interpolation is used, but in the reverse way; the procedure follows:

(1) In the table, find the two mantissas between which the given mantissa lies, and write the three figures corresponding to the smaller of these two mantissas.

(2) Next find the difference between these two consecutive mantissas, and also the difference between the smaller of them and the given mantissa whose antilog we are seeking.
(3) Divide the latter difference by the former, carrying the division to the nearest tenth.
(4) Annex the figure so found to the three figures already found, making it the fourth figure of the antilogarithm.
(5) Locate the decimal point in the antilogarithm by inspection of the given logarithm.

It sounds far more difficult than it really is.

EXAMPLE 1: Find the antilog of 2.3942.

SOLUTION: The given mantissa .3942 lies between 3927 and 3945; thus the first three figures of the antilog are "247."

$$\begin{array}{r} .3945 \\ .3927 \\ \hline .0018 \end{array} \qquad \begin{array}{r} .3942 \\ .3927 \\ \hline .0015 \end{array} \qquad \frac{.0015}{.0018}=.8$$

hence antilog=247.8, *Ans.*

EXAMPLE 2: Find the antilog of 7.5029—10.

SOLUTION: Antilog lies between 318 and 319.

$$\begin{array}{r} .5038 \\ .5024 \\ \hline .0014 \end{array} \qquad \begin{array}{r} .5029 \\ .5024 \\ \hline .0005 \end{array} \qquad \frac{.0005}{.0014}=.4$$

hence antilog=.003184, *Ans.*

Exercise 49.

Find the number whose logarithm is:

1. 1.3806	5. 1.3989	9. 2.4416
2. .6432	6. 9.3001—10	10. 4.7215
3. 2.4237	7. .7625	11. 7.9103—10
4. 9.6145—10	8. 8.4182—10	12. 3.0462

Laws of Exponents Applied to Logarithms. By applying the laws of exponents to logarithmic expressions, we find that if M and N are any two numbers, then:

Law I. $\log(MN)=\log M+\log N$.

Suppose we let $\log M=x$,
and $\log N=y$.
then $10^x=M$, and $10^y=N$.
But $(MN)=(10^x)\cdot(10^y)=10^{(x+y)}$;
hence $\log(MN)=x+y$,
or $\log(MN)=\log M+\log N$.

In other words, *the logarithm of a product equals the sum of the logarithms of its factors.*

For example: $\log 21 = \log 7 + \log 3$
$\log 5000 = \log 1000 + \log 5$
$\log (372 \times 28.1) = \log 372 + \log 28.1$
$\log Prt = \log P + \log r + \log t$

NOTE: This property of logarithms, viz.
$\log (PQ) = \log P + \log Q,$
should not be confused with the following:
$\log P + \log Q \neq \log (P+Q).$

In short, the *sum of two logarithms does not* equal the logarithm of their sum; neither does the logarithm of the *sum of two numbers* equal the sum of their logarithms.

Law II. $\log\left(\frac{M}{N}\right) = \log M - \log N$

That is, *the logarithm of a quotient equals the logarithm of the numerator diminished by the logarithm of the denominator.*

NOTE: This should not be misinterpreted; $\log (M-N) \neq \log M - \log N$, and $\frac{\log M}{\log N} \neq \log M - \log N$. In short, the logarithm of the difference of two numbers is the logarithm of *their* difference, not the difference of their logarithms. Again, the ratio of two logarithms is simply a number; it is not the same as the logarithm of a ratio, and is therefore not equal to the difference of two logarithms.

Law III. $\log (N)^p = p(\log N)$

Law IV. $\log(N)^{\frac{1}{p}} = \log\sqrt[p]{N} = \frac{1}{p}(\log N)$

Thus, for example, we have:

$\log r^2 = 2 \log r$
$\log v^3 = 3 \log v$
$\log \sqrt{d} = \frac{1}{2} \log d$
$\log \sqrt[3]{k} = \frac{1}{3} \log k$
$\log p^{3/2} = \frac{3 \log p}{2}$
$\log \sqrt{\pi r^2} = \frac{1}{2} \log \pi + \log r$

Multiplying and Dividing by Using Logarithms. The chief practical value of logarithms is in shortening the multiplication and division of numbers. When skill in their use has been achieved, the time and labor required by longhand multiplication and division in extensive computations is very materially lessened, as the following examples will show.

EXAMPLE 1: Multiply $396.2 \times 8.735 \times .07434$.

SOLUTION:

$$\begin{aligned} \log 396.2 &= 2.5979 \\ \log 8.735 &= .9413 \\ \log .07432 &= \underline{8.8712-10} \\ & \quad 12.4104-10 \end{aligned}$$

antilog $2.4104 = 257.3$, *Ans.*

EXAMPLE 2: Find the value of

$$\frac{.376 \times 4.018 \times 57.64}{.0675 \times 3.1416}.$$

SOLUTION:

$$\begin{aligned} \log .376 &= 9.5752-10 \\ \log 4.018 &= .6040 \\ \log 57.64 &= \underline{1.7607} \\ & \quad 11.9399-10 \\ & \quad \underline{9.3264-10} \\ & \quad 2.6135 \end{aligned}$$

$$\begin{aligned} \log .0675 &= 8.8293-10 \\ \log 3.1416 &= \underline{.4971} \\ & \quad 9.3264-10 \end{aligned}$$

antilog $2.6135 = 410.8$, *Ans.*

Exercise 50.

Find, by using logarithms, the value of:

1. $\dfrac{622.5 \times .0317}{14.8}$
2. $\dfrac{.00905}{2.54 \times 3614}$
3. $\dfrac{46.22 \times 309.4}{162 \times .084}$
4. $2 \times 22/7 \times 18.6 \times 35.8$
5. $1/365 \times .0525 \times 1650$
6. $\dfrac{26.8 \times .00048 \times 127.5}{709.2 \times .486}$

Using Logarithms to Find Roots and Powers. Another extremely practical use of logarithms is in finding roots and powers of numbers.

EXAMPLE 1: Find $\sqrt[5]{62.4}$.

SOLUTION: Let $x = (62.4)^{1/5}$

$\log x = 1/5 (\log 62.4)$

$\log x = (1/5)(1.7952) = .3590$

$x = 2.286$, *Ans.*

EXAMPLE 2: Compute the value of $(1.055)^{12}$.

SOLUTION: Let $x = (1.055)^{12}$

$\log x = 12 (\log 1.055)$

$\log x = (12)(.0232) = .2784$

$x = 1.898$, *Ans.*

EXAMPLE 3: What is the value of $x = \dfrac{.0046 \times \sqrt[3]{85.7}}{(.063)^2}$.

SOLUTION: $\log .0046 = 7.6628-10$

$\frac{1}{3}\log 85.7 = .6443$

$8.3071-10$

$2\log .063 = 17.5986-20$

$\log x = .7085$

$x = 5.111$, *Ans.*

EXAMPLE 4: In the formula $t=\sqrt{\frac{2s}{g}}$, find t when $s=1155$ and $g=32.2$.

SOLUTION: $t=\left(\frac{2s}{g}\right)^{\frac{1}{2}}$

$\log t = \frac{1}{2}[\log 2+\log s-\log g]$

$= \frac{1}{2}[\log 2+\log 1155-\log 32.2]$

$\log 2 = .3010$

$\log 1155 = 3.0626$

3.3636

$\log 32.2 = 1.5079$

$2\overline{)1.8557}$

$\log t = .9279$

$t = 8.47$, *Ans.*

Exercise 51.

Using logarithms, compute each of the following:

1. $\sqrt[3]{756}$
4. $\sqrt[3]{9.47}$
7. $(1.045)^9$
2. $\sqrt[5]{895}$
5. $(6.29)^{\frac{2}{3}}$
8. $(40.21)^3$
3. $(.562)^5$
6. $\sqrt[4]{24.38}$
9. $\sqrt{.000394}$
10. $(8.9)^3\times(27.84)$
11. $\sqrt[3]{162.4}\times(85.36)$
12. $(.364)^2\times 29.25\times\sqrt[5]{162.4}$
13. $\sqrt[3]{\frac{215.2}{3.14\times 4.57}}$
14. The formula for the volume of a cylinder is $V=\pi R^2H$; find R when $V=906.0$, $H=14.6$ and $\pi=3.142$.
15. The volume of a sphere is given by $V=\frac{4}{3}\pi R^3$; find to the nearest hundredth the diameter of a sphere whose volume is 85.4 cu. in., using $\pi=3.142$.

Solving Exponential Equations by Use of Logarithms. Still another useful application of logarithms is in connection with the solution of *exponential equations,* i.e., an equation in which the variable (or unknown quantity) is an exponent. Such equations frequently arise in engineering and technical problems.

EXAMPLE 1: Solve for x: $18=(4.5)^x$.

SOLUTION: $\log 18=(x)(\log 4.5)$

$$x=\frac{\log 18}{\log 4.5}=\frac{1.2553}{.6532}=1.92,\ \textit{Ans.}$$

EXAMPLE 2: Solve: $5^k=10^{k+1}$.

SOLUTION: $k(\log 5)=(k+1)(\log 10)$

$$\log 5=.6990$$
$$\log 10=1.0000$$
$$.699k=k+1$$
$$k=-\frac{1}{.301}=-3.33,\ \textit{Ans.}$$

Exercise 52.

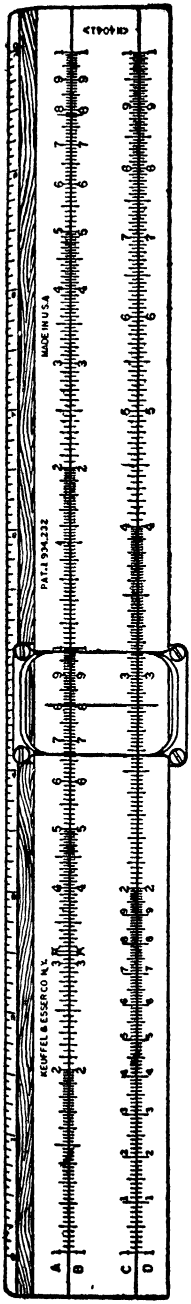
Solve the following exponential equations:

1. $5^x=18$
2. $1.5=(1.03)^x$
3. $12.8=75^{x_2}$
4. $(3.02)^x=100$
5. $(3.4)^x=22.84$
6. $(2^x)(10^x)=30$
7. $\sqrt[x]{12}=4$
8. $(1.04)^{t+1}=6.8$
9. A principal of $400 at compound interest for a certain length of time at 3½% a year, compounded annually, amounted to $750. Find the time.
10. A machine originally costing $1800 depreciates in value each year 15% of its value at the beginning of the year. What is its value at the end of 6 years? [*Hint:* $V=1800\ (.85)^6$]. In how many years will it be worth $900?

12. THE SLIDE RULE

Mechanical Computation. Historically speaking, man has made use of various sorts of mechanical aids to computation throughout the ages. The slide rule is one such mechanical device. It was invented some 300 years ago by an English mathematician, William Oughtred; subsequently improved, in its modern form it was invented about 1850 by Amédée Mannheim, and is now known by his name. In addition to the simple Mannheim rule, there are available today some ten or twelve other types, more elaborate and more complicated, and designed for a variety of mathematical computations as well as for special purposes. In the present discussion we shall confine our attention to the simple Mannheim rule.

Description of the Rule. Generally made of wood, xylonite or celluloid, the slide rule is from 5 to 10 inches in length. It consists of three parts: the rule, which is grooved; the slide, which is carefully fitted so that it slides easily in the grooved rule from left to right and vice versa; and the hairline runner. The slide and the rule are both faced with accurately graduated scales.



In all, there are four rows of figures on the front of the slide rule:

Row A: two complete logarithmic scales, on the rule, giving directly squares and square roots.

Row B: two complete logarithmic scales, exactly like A, but on the slide.

Row C: a single logarithmic scale, also on the slide.

Row D: another single logarithmic scale, exactly like C, but on the rule.

On the back of the slide there are to be found three more scales:

Row S: a trigonometric scale of sines, used in conjunction with the A and B scales.

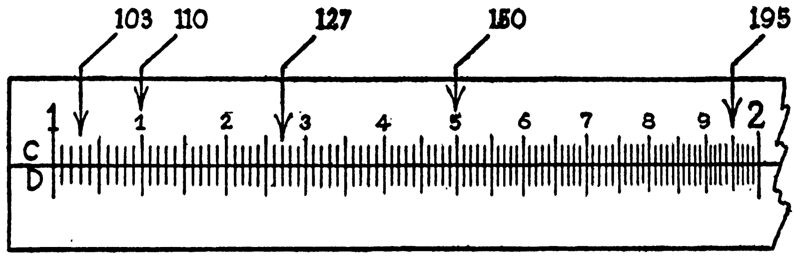
Row T: a trigonometric scale of tangents, used in conjunction with the C and D scales.

Row L: a scale of equal parts, used to find the common logarithms of numbers.

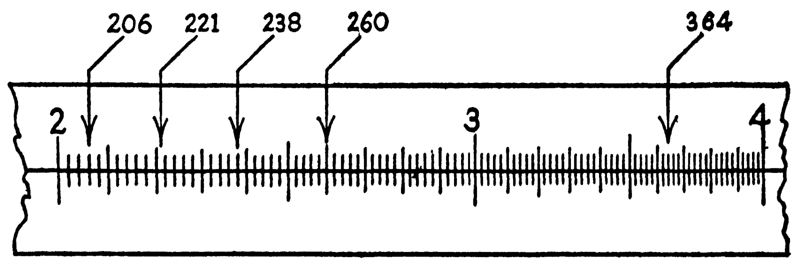
The C and D Scales. Confining ourselves to the front of the slide rule, and noting the two scales marked C and D, respectively, we find that the graduations in each case begin with 1 at the extreme left and are numbered 1, 2, 3, 4, etc., from the left to 8, 9, 1 at the right. The reason that the figure at the extreme right is 1 and not 10 is that *all* the figures on the scales, as marked, are *arbitrary.* Thus the initial 1 at the left index may represent 10, 100, 0.1, 0.01, etc., depending upon the particular computation, but once the value of the left index has been chosen, the same ratio must be observed throughout; e.g., if we start reading it as 10, then the other figures are to be read 20, 30, 40, etc.; if we commence with .01, then the other figures are read as .02, .03, .04, etc. It will also be observed that the *actual* space or distance from 1 to 2 is the same as that from 2 to 4, and also as that from 4 to 8; that is what makes it a *logarithmic* scale.

The segment between 1 and 2 is subdivided into 10 parts, each succeeding part decreasing slightly in size; each of these 10 subdivisions is again divided into 10 smaller divisions. If space permitted, these subdivisions would be carried out all along, but, because of their decreasing size, they are later subdivided only into halves or fifths, which must be interpreted as decimal divisions and not as fractional ones. Nor are they all marked with numerals, since this would tend to overcrowd the rule and make it difficult to read; moreover, as one acquires skill and facility in using the slide rule, these additional numbers are not really needed at all.

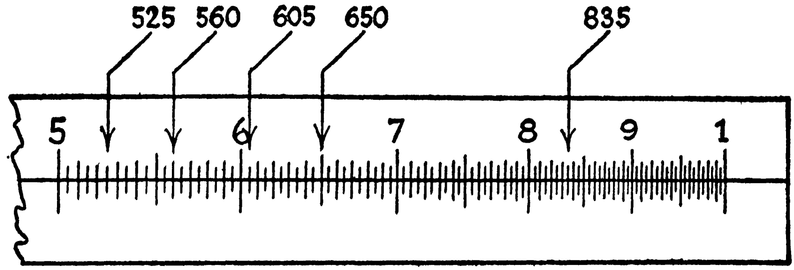
To learn to read these scales, study the following sketches given of portions of the rule and slide, noting especially the "sample readings" indicated.



It will be seen that this portion of the C and D scales (between 1 and 2) indicates accurately numbers which contain three digits, if the first digit is 1. If a number contains four digits (the last one not being zero), the number can be read accurately to three digits, but the fourth digit must be estimated. Do not be concerned about this; *always estimate to the maximum degree the smallest subdivision when reading a slide rule.*



Turning our attention next to the portion of the C and D scales between 2 and 4, it will be shown how these figures are read. Any number beginning with 2 and 3 is read from this part of the scale. In this portion of the scale each large unnumbered division represents 1/10 of 100, or 10; furthermore, since there are five unnumbered divisions between any two large unnumbered divisions, each small unnumbered division represents 2. Thus all even numbers between 200 and 400 can be read accurately, but all odd numbers must be estimated.



Finally, the portion of the C and D scale from 4 to 10 is used to locate any number whose first digit is 4, 5, 6, 7, 8 or 9. The first two digits can always be read accurately; the third digit can be read accurately only if it is a 5, otherwise it must be estimated.

The Principle of Proportion. The slide rule is not designed for addition and subtraction; its unique adaptability is for multiplication, division, proportions, powers and roots. It is a basic principle of the slide rule that, no matter where the slide is placed, *all* the numbers on the slide bear the same ratio to their corresponding or coinciding numbers on the rule. This holds true of the C and D scales, and also of the A and B scales. For example, pull the slide out to the right until 1 on C corresponds exactly with, or is "over," 2 on D; you will then note that the ratio 1:2 exists between every pair of coinciding numbers on the C and D scales, respectively. Thus if we set the first two terms of a proportion against each other on the slide and the rule, we find the third term on the slide coinciding with the fourth term on the rule. Hence:

$$\begin{pmatrix}\text{any number}\\ \text{on C}\end{pmatrix} : \begin{pmatrix}\text{the number}\\ \text{under it on D}\end{pmatrix} = \begin{pmatrix}\text{any other}\\ \text{number on C}\end{pmatrix} : \begin{pmatrix}\text{the number}\\ \text{under it on D}\end{pmatrix}$$

This principle of proportion can also be expressed as follows, giving a rule of procedure for using the slide rule to solve for the fourth term of a proportion:

C	set first term	under third term
D	over second term	find fourth term

EXAMPLE: Solve for x: $\dfrac{3}{8\frac{1}{2}} = \dfrac{15}{x}$.

SOLUTION:

C	set 3	under 15	*Ans.*, 42.5
D	over 8.5	find 42.5	

Multiplication. Since the multiplication of two factors, say 4×7, is the same as finding x in the proportion $1:4=7:x$, the rule for multiplication on the slide rule is simply this:

C	set 1	under the other factor
D	over one factor	find their product

EXAMPLE 1: Multiply 14×6½.

SOLUTION:

C	set 1	under 65
D	over 14	find 91

Ans., 91

NOTE: The location of the decimal point in the answer is most readily achieved by inspection of the original numbers rather than by any other rule; thus it is obvious that 14×6½ is approximately equal to 100, so that when reading the value "9—1" it would be read as 91 rather than as 9.1 or 910.

EXAMPLE 2: Find the product of 5¼×24.

SOLUTION:

C	set 1	under 24
D	over 525	find 126

Ans., 126

NOTE: If when drawing the slide out *to the right* in order to set 1 on C over one of the factors on D the other factor on C extends beyond the right terminal 1 on D, then the slide is drawn to the left and the right terminal 1 on C is set over the first factor on D.

EXAMPLE 3: Multiply: 17.6×.0428.

SOLUTION:

C	set 1	under 428
D	over 176	753

Ans., 0.753

NOTE 1: Since 18×.04=.72, the decimal point is placed before the 7 in reading "753."

NOTE 2: In setting 428, the terminal "8" is *estimated* as nearly as possible between "425" and "430"; likewise, when reading the product "753" on the D scale, the terminal "3" is also estimated as closely as possible. In this case, the product might have been read as .752 or .754 instead of .753, which would be an error of only one part in a thousand; the actual product is .75328. But since the factors are only given to three significant figures each, the product is adequately expressed as .753.

Division. The procedure for dividing one number by another is quite similar, as might be expected, since division is the reverse of multiplication. Thus the rule is:

C	set divisor	under 1
D	over dividend	find quotient

EXAMPLE 1: Divide 64 by 4½.

SOLUTION:

C	set 45	under 1
D	over 64	find 1425

Ans., 14.25

EXAMPLE 2: Find 0.256÷33.2.

SOLUTION:

C	set 332	under 1
D	over 256	find 771

Ans., 0.00771

EXAMPLE 3: Divide 80.3 by .049.

SOLUTION:

C	set 49	under 1
D	over 803	find 1639

Ans., 1639

Combined Multiplication and Division. Here the extreme utility of the slide rule becomes even more evident, as the following illustrations will make abundantly clear.

EXAMPLE 1: Find: 18×4.2×.17.

SOLUTION:

C	set 1	bring runner to 42	bring 1 to runner	under 17
D	on 18			find 1285

Ans., 12.85

EXAMPLE 2: Find: $\dfrac{27.2\times 44\times .88}{.73\times 6.4}$

SOLUTION:

C	set 73	runner to 44	64 to runner	under 88
D	on 272			find 2255

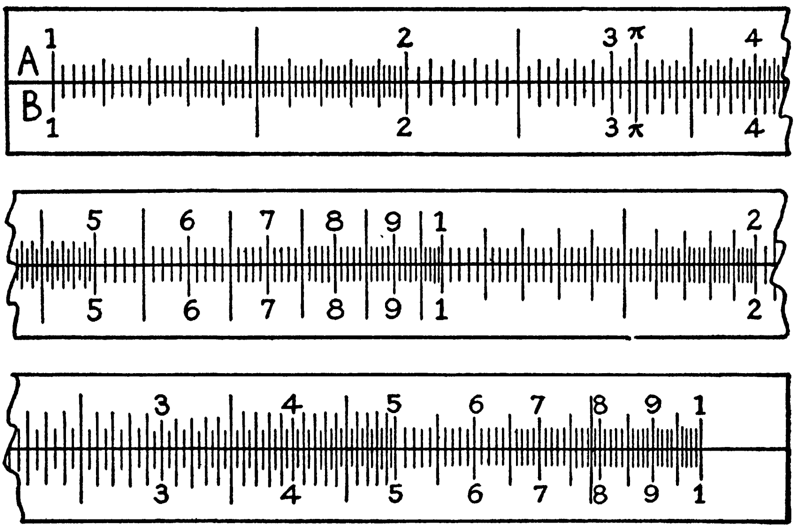
Ans., 225.5

Exercise 53.

Using the slide rule, find the following:

1. 39.6×24
2. 87×6.5
3. 6.21×9.24
4. 7.4÷32
5. 60.5÷0.49
6. 4.62×.866×29
7. 3.08×144×1.6
8. 28.5×3.14×0.045
9. 835÷21.8
10. .32÷0.077
11. $\dfrac{63\times 22}{17.5}$
12. $\dfrac{8.02\times 30.6}{68\times 2.5}$
13. $\dfrac{46.5\times 3.14}{80.2\times .35}$
14. $\dfrac{.707\times 18.4}{2.54}$

Squares and Square Roots. Upon examining the upper scales A and B, it will be found that all the numbers on both the A and B scales are the *squares* of their coinciding numbers on the C and D scales; conversely, all the numbers on the C and D scales are the *square roots* of their coinciding numbers on the A and B scales. Thus, if the runner is set to 3 on the D scale, the hairline coincides with 9 on the A scale; this may



be read as $3^2=9$, or $\sqrt{9}=3$; also as $30^2=900$, or $\sqrt{900}=30$; also as $300^2=90000$, or $\sqrt{90000}=300$; etc. Furthermore, if the runner is set to 5 on the D scale, the hairline will coincide with 25 on the A scale (not 2.5); i.e., $5^2=25$, or $\sqrt{25}=5$; also, $50^2=2500$, or $\sqrt{2500}=50$; etc. It should further be observed that the A scale is not graduated in the same manner as the D scale, since the A scale consists of *two parts,* the left- and right-hand scales, each running from "1" to "1." But if the left index of the left half of the A scale is "1," then the right terminal index of that half of the scale is "10," and the "2" in the right half of the A scale is then "20," etc.

In other words, to find the square root of a number, we locate that number on the A scale, observing the following rules; then set the runner on the number (on the A scale) whose root is to be found, and under it, on the D scale, read the desired square root:

Rule 1: To find the square root of a number having an *even* number of digits to the left of the decimal point, use the *right half* of the A scale; e.g., to find $\sqrt{62}$, $\sqrt{38.4}$, $\sqrt{6531.5}$, etc.

Rule 2: To find the square root of a number having an *odd* number of digits to the left of the decimal point, use the *left half* of the A scale; e.g., to find $\sqrt{6}$, $\sqrt{182}$, $\sqrt{3.1416}$, $\sqrt{856.2}$, etc.

Rule 3: To find the square root of a decimal having an *even* number of zeros to the right of the decimal point, use the *right half* of the A scale; e.g., $\sqrt{0.35}$, $\sqrt{0.0042}$, $\sqrt{0.831}$, $\sqrt{0.0000562}$, etc.

Rule 4: To find the square root of a decimal having an *odd* number of zeros to the right of the decimal point, use the *left half* of the A scale; e.g., $\sqrt{0.0269}$, $\sqrt{0.00048}$, $\sqrt{0.086}$, etc.

EXAMPLE 1: Find the square root of 39.5.

SOLUTION: Since 39 contains an even number of digits to the left of the decimal point, we use the *right half* of the A scale. Set runner on 395 on right side of A scale; under hairline on D scale read "628."

Ans., $\sqrt{39.5}=6.28$.

EXAMPLE 2: Find $\sqrt{431}$.

SOLUTION: Since 431 contains an odd number of digits to the left of the decimal point, we use the *left half* of the A scale. Set runner on 431 on left side of A scale; under hairline on D scale read "2075."

Ans., $\sqrt{431}=20.75$.

EXAMPLE 3: Find $\sqrt{0.88}$.

SOLUTION: Since the decimal has no zeros, i.e., an even number ot zeros, to the right of the decimal point, we use the *right half* of the A scale. Set runner on 88 on right side of A scale; under hairline read "938."

Ans., $\sqrt{0.88}=0.938$.

EXAMPLE 4: Find $\sqrt{0.0825}$.

SOLUTION: Since the decimal has an odd number of zeros to the right of the decimal point, we use the *left half* of the A scale. Set runner on 825 on left side of A scale; under hairline read "287."

Ans., $\sqrt{0.0825}=0.287$.

Exercise 54.

Find the square root of each of the following by using the slide rule:

1. 44	5. 8.25	9. 3450
2. 24.8	6. .846	10. 0.005
3. 70.5	7. 0.00032	11. 3.07
4. 137	8. 0.0643	12. 0.0076

SELECTED REFERENCES FOR FURTHER STUDY

General Mathematics

BRENEMAN, J. W. Mathematics. McGraw-Hill Book Co.
BROWN, W. L. Related Mathematics. J. Wiley & Sons.
DOUGLASS, H. R. & KINNEY, L. B. Everyday Mathematics. H. Holt & Co.
KANZER, E. M. & SCHAAF, W. L. Essentials of Business Arithmetic. D. C. Heath & Co.
LASLEY, S. J. & MUDD, M. F. The New Applied Mathematics. Prentice-Hall, Inc.
LENNES, N. J. Senior Practical Mathematics. Macmillan Co.
MUELLER, C. H. Geometric Concepts. J. Wiley & Sons.
SCHAAF, W. L. Mathematics for Everyday Use. Garden City Publishing Co.
STONE, J. C. & MALLORY, V. S. Mathematics for Everyday Use. Benj. H. Sanborn & Co
VAN TUYL, G. H. Mathematics at Work. American Book Co.

CHAPTER III

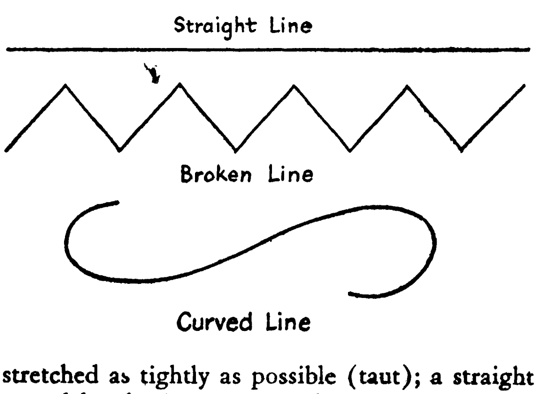
PRACTICAL GEOMETRY

William L. Schaaf

13. LINES AND ANGLES

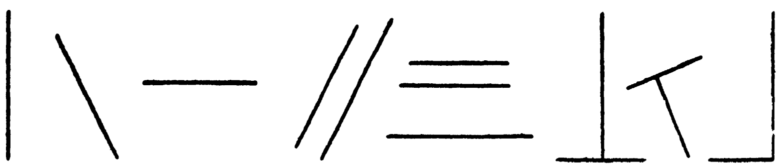
Nature of Geometry. The subject of geometry may be studied in two different ways—either as a system of logical demonstrations, or as a body of practical facts and relationships concerning geometric figures and forms. In this book geometry will be presented from the second point of view, since it is the practical applications of geometric principles with which the mechanic and vocational student needs to be familiar. Scarcely a single shop operation or trade process will be encountered which does not involve some geometric relationship or formula. These basic geometric relations can readily be learned without the conventional logical treatment. Most of them are simple to understand on the basis of common sense (intuition) and practical measurements—especially if approached through everyday illustrations from shop problems.

Lines and Points. Geometrically we think of a *point* simply as representing a certain position only; it has no magnitude. A *line* may be regarded as the path of a moving point, or the series of positions traced by a point in motion; it has length only—no width or thickness. Lines are either straight, curved, or broken, or a combination of these. A *straight* line may be thought of as an "imaginary" string stretched as tightly as possible (taut); a straight line may also be represented by the intersection of two plane surfaces, such as the edge of a



cube. Any line that is not straight is *curved,* unless, of course, it consists of a series of connected straight lines, i.e., a *broken* line.

If the length of a straight line is not specified, but only its direction, it is to be thought of as extending indefinitely in both directions. If, on the other hand, its terminal, or end points, are specified, it is more properly called a *line segment,* or simply a segment. The various positions in which



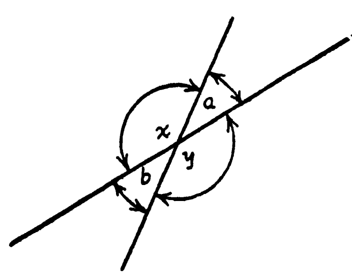
Vertical line | Oblique line | Horizontal line | Oblique Parallels | Horizontal Parallels | Perpendicular lines

straight lines may lie are indicated by familiar names, as suggested in the accompanying figure. If the word "line" is used without an adjective before it, the reader is to assume that a straight line is meant.

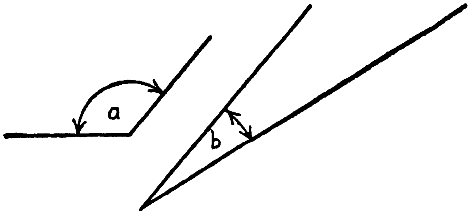
Parallel lines are two or more lines, all lying in the same plane surface, and which never meet however far they are extended (prolonged) in either direction. Any two parallel lines are equally distant from each other throughout their entire length; in a series of three or more parallels, however, some may be closer to one another than others.

Perpendicular lines are lines forming right angles with each other, where by a right angle is meant a quarter of a complete rotation. A vertical line (plumb line) meeting a horizontal line is an illustration of two perpendiculars; perpendicular lines need not, however, be horizontal and vertical, as shown above. When two lines form right angles with each other, each line is said to be perpendicular to the other.

It simply remains to point out that the shortest distance between any two specified points is a straight line joining those points. Also, that two straight lines can intersect in one point only; indeed, a point might be described as the *position* where two lines intersect. When two straight lines intersect, two pairs of equal angles are formed; the opposite angles, called *vertical angles,* are equal to each other; i.e., $\angle a = \angle b$, and $\angle x = \angle y$.



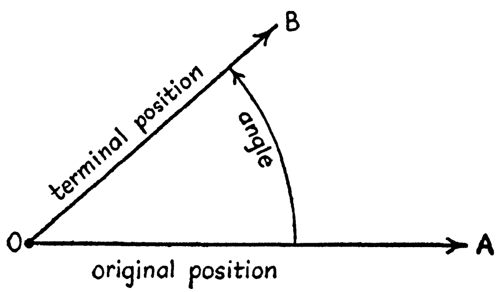
Angular Measurement. An *angle* may be described as the "opening" between two lines that intersect or meet. The lines are called the *sides* of



Angle a is larger than angle b, even though its sides are shorter.

the angle, and the point where they intersect or meet is called the *vertex* of the angle. The *size* of the angle in no way depends upon the length of its sides, but only upon the "amount" of opening or divergence between them.

Another way of thinking of an angle is as follows. If a line is rotated about any fixed point on the line, thus taking the line out of its original position to a new terminal position, the two positions are said to form an angle with each other; the point around which the rotation took place is the vertex of the angle. In other words, an angle is really a certain amount of *rotation* or turning. Thus an angle cannot be said to have length, or width, or area, etc., but simply rotation.



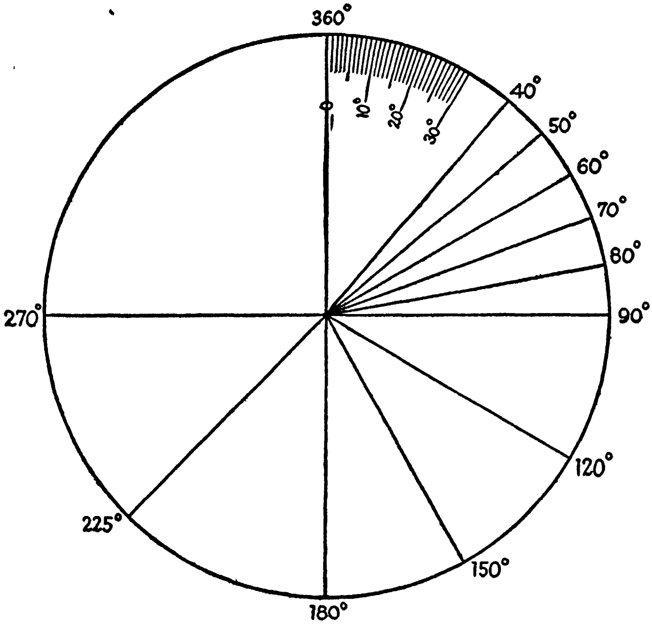
If a line is rotated until it returns to its original position, i.e., through one complete revolution, it is said to have described or formed a *round angle,* or to have turned through 360 degrees. If we consider any particular point on a line, except the pivotal or turning point, it will have described a *circle* when the line has made one complete revolution. Thus there are 360 degrees of angle in every circle, regardless of the size of the circle. For convenience, the standard unit of angular measurement is the *degree,* or 1/360 of a complete revolution.

Degrees, Minutes and Seconds. Angles are measured in terms of the following units:

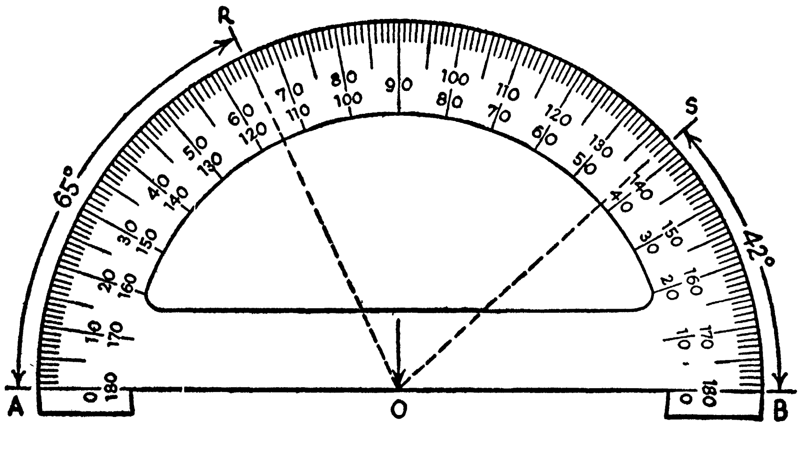
TABLE OF ANGULAR MEASUREMENT

60 seconds	=1 minute
60 minutes	=1 degree
90 degrees	=1 quadrant
360 degrees	=1 circumference

Degrees are designated by (°); minutes by (′); and seconds by (″); thus an angle of 32 degrees, 45 minutes, 20 seconds is written as 32°45′20″. The accompanying diagram shows a circle divided into degrees; each of the smallest subdivisions represents 1°.

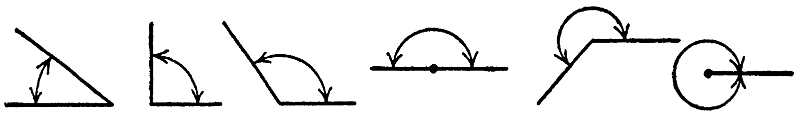


The Protractor. This is the instrument commonly used to lay out or measure angles. There are two sets of scales, each from 0° to 180°, one running clockwise from A to B, the other counterclockwise from B to A. Whenever an angle is to be measured or laid out, the vertex is always put



at point 0 on the protractor. The figure shows how an angle AOR (65°) and another angle BOS (42°) can be measured or laid out with the aid of a protractor.

Kinds of Angles. A *right* angle is ¼ of a revolution, or 90°. An *acute* angle is an angle less than a right angle, or one containing less than 90°; an *obtuse* angle is one that is greater than a right angle, or containing

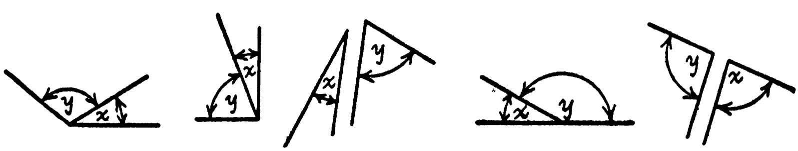


Acute Angle (a) | Right Angle (b) | Obtuse Angle (c) | Straight Angle (d) | Reflex Angle (e) | Round Angle (f)

more than 90° (but less than 180°). If a line has been rotated through *half* a revolution, or 180°, it is said to form a straight angle; thus any straight line may be regarded as a "straight angle" with its vertex at any point on the line we may wish to choose. Again, if a line is rotated through more than 180° but less than 360°, it is said to form a *reflex* angle; however, when two lines form angles as in (e), *unless otherwise specified,* "the angle" between them is the angle *less than* 180°, rather than the reflex angle ("reflex" means, literally, "bending back"). Finally, a complete revolution of 360° is sometimes called a *round angle;* similarly a line making two successive complete revolutions is said to have described an angle of 720°; three revolutions, 1080°; 1½ revolutions, 540°; etc.

Related Angles. A few common relations between angles should be understood. Vertical angles, always occurring in pairs, have already been mentioned; such angles are equal to each other (oppositely) in pairs. Other important relations are as follows:

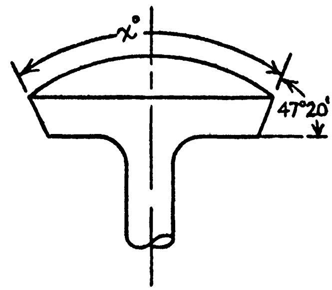
(1) All right angles are equal.
(2) A straight angle equals two right angles.
(3) Two angles are *adjacent* if they have a side in common.
(4) Two angles are *complementary* if their sum equals 90°. Each is the complement of the other; they may or may not be adjacent.
(5) Two angles are *supplementary* if their sum equals 180°. Each is the supplement of the other; they need not be adjacent.



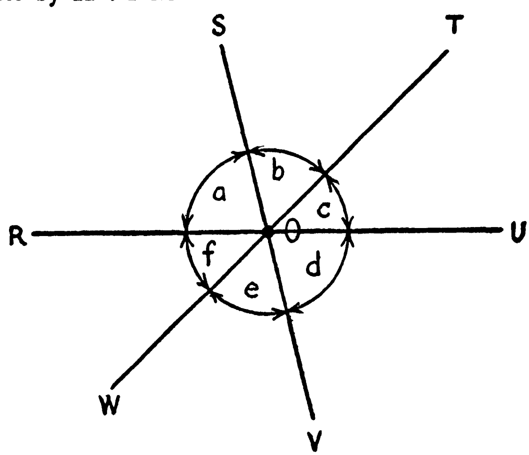
Adjacent Angles | Complementary Angles | Supplementary Angles

Exercise 55.

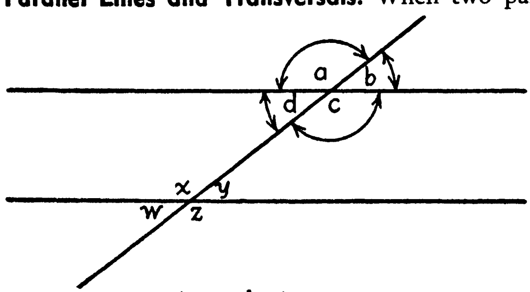
1. What is the complement of 39°? of 22½°? of 44°20′? of 67°41′12″?
2. What is the supplement of 32°? of 45°? of 90°? of 122°? of 57°29′? of 160°44′48″?



3. Find the angle x of the valve seat of the gasoline engine valve shown.
4. If ∠ROT=140°, and ∠TOV=120°, find the number of degrees in ∠SOU.
5. If ∠WOS=140°, and ∠ROT=125°, find ∠d.
6. Two angles are complementary. The greater exceeds the less by 22°. Find the angles.
7. Find the number of degrees in an angle the sum of whose supplement and complement is 202°.
8. The supplement of a certain angle exceeds three times its complement by 10°. Find the angle.



Parallel Lines and Transversals. When two parallel lines are cut by a third line, the latter, which is said to be a *transversal,* forms four pairs of angles with the two parallels. These are known as follows:



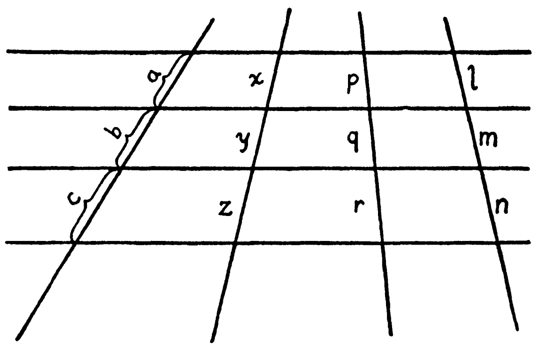
∠a and ∠x
∠b and ∠y
∠c and ∠z
∠d and ∠w
} are *corresponding* angles.

$\angle d$ and $\angle y$ } are *alternate-interior* angles.
$\angle c$ and $\angle x$

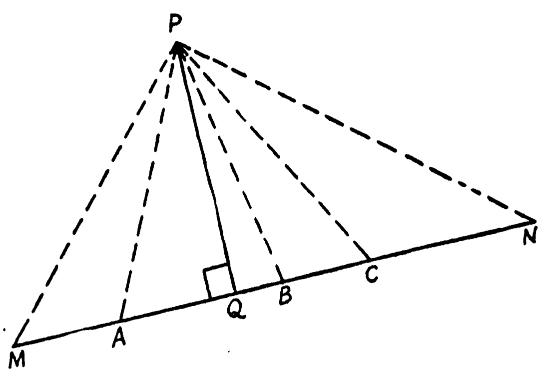
$\angle a$ and $\angle z$ } are *alternate-exterior* angles.
$\angle b$ and $\angle w$

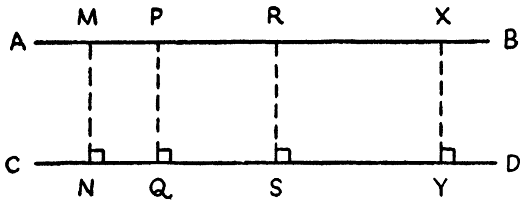
By studying the figure carefully, it will be seen that whenever a transversal intersects a pair of parallel lines, all the pairs of corresponding angles, as well as all the pairs of alternate angles, whether "interior" or "exterior," are equal. Thus: $\angle a = \angle x$; $\angle d = \angle y$; $\angle a = \angle z$; etc. Angles that lie on the *same* side of the transversal and are *included* between the parallels, are not equal, but *supplementary* to each other; thus $\angle d + \angle x = 180°$, and $\angle c + \angle y = 180°$.

If a series of parallel lines is cut by two or more transversals, and the segments cut off on any one of these transversals are equal to each other, then the segments on every other transversal are also equal to each other. On the diagram, this means that if $a=b=c$, then $x=y=z$; $p=q=r$; and $l=m=n$. It does *not* mean, however, that $a=x=p=l$, or that $b=y=q=m$, etc.; whether this is true depends upon the angles at which any two transversals cut the parallels.



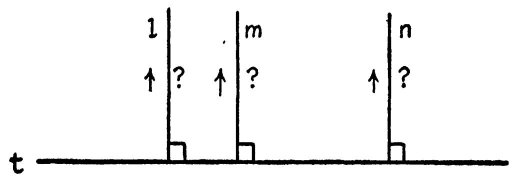
Meaning of Distance. In geometry, when we speak of the *distance* from a point to a line (or from a line to a point) it is understood to mean the *shortest* distance, or the *perpendicular* drawn from the point to the line. Thus, while PM, PA, PB, PC, and PN do represent "distances" from P *to various points on MN*, the distance from P *to the line MN* is PQ, and no other. Similarly, the distance between two parallel lines refers to the *perpendicular* distance be-



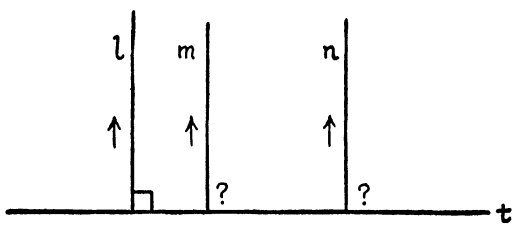


tween them, anywhere along the line; this perpendicular distance is the same at any point, since two parallel lines are everywhere equidistant.

Further Properties of Parallels and Perpendiculars. The following relationships are not only fundamental, but frequently quite useful as well.

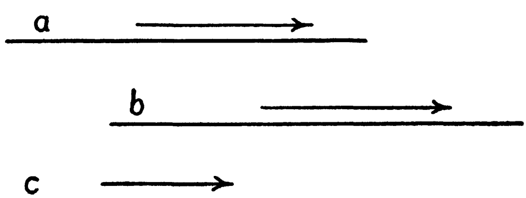


(1) If two or more lines are all perpendicular to another line, then they are parallel to one another. For example, if *l*, *m*, and *n* are each $\perp$ *t*, then *l*, *m* and *n* are $\|$ each other.



(2) If any one of two or more parallel lines is perpendicular to another line, then all of them are perpendicular to that line. For example, if *l*, *m* and *n* are $\|$ each other, and $l \perp t$, then *m* and *n* are also $\perp$ *t*.

(3) If a line is perpendicular to one of two or more parallels, then it is perpendicular to all of them. For example, if *l*, *m*, and *n* are $\|$, and if $t \perp l$, then *t* is also $\perp$ *m* and *n*.

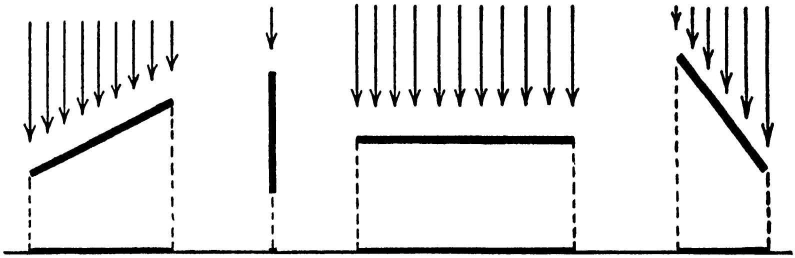


(4) If two lines are both parallel to a third line, then they are parallel to each other. For example, if $a \| x$, and $b \| x$, then $a \| b$.

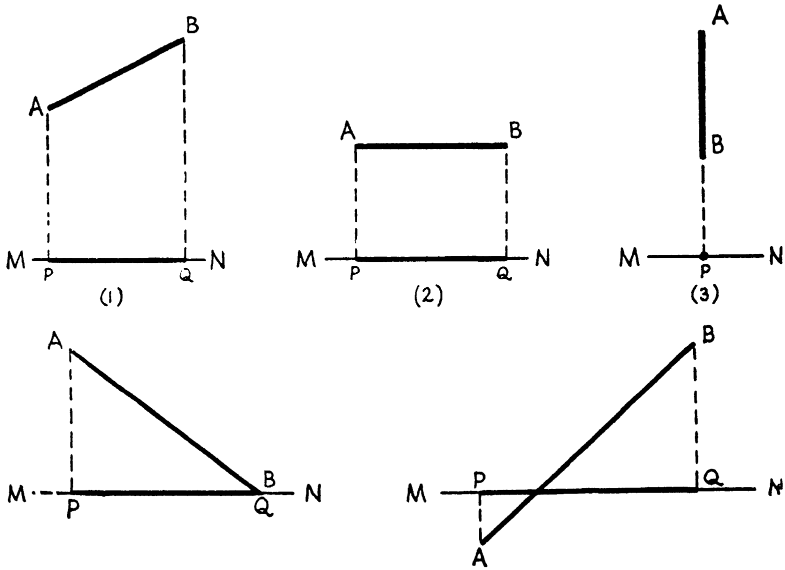
(5) If a line is parallel to one of two other parallel lines, then it is parallel to the other. For example, if $a \| b$, and $x \| a$, then *x* is also $\|$ *b*.

Projections. Everybody knows that when the sun is directly overhead, the shadow cast by a stick held obliquely will be shorter than the stick. The more nearly vertically it is held, the shorter the shadow becomes;

when it is exactly vertical, its shadow is shortest and approximates a point. The more nearly horizontal its position, the longer its shadow be-

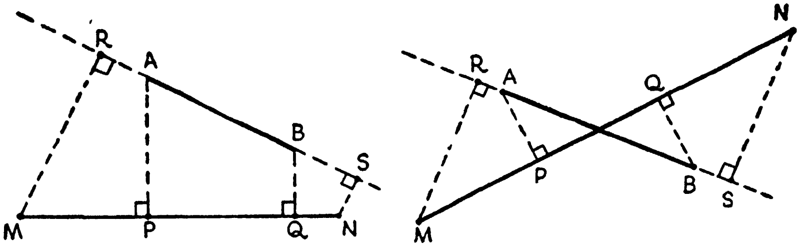


comes; when exactly horizontal, the shadow is longest, being equal to the length of the stick itself. In all these cases, the shadow of the stick is called its *projection* (technically, its orthographic projection). Or, expressed somewhat differently, the projection of one line upon a second line is the segment included between the feet of the perpendiculars drawn to the second line from the extremities of the first line. This is illustrated in the figures below:



In each case, the projection of line AB upon line MN is the segment PQ; even in case (3), where the projection is the point P, this point may be regarded as a segment PQ of zero length (the points P and Q having "come together," or coincided).

Furthermore, given any two lines, either line may be projected upon the other, although one of them may have to be prolonged first. Thus in

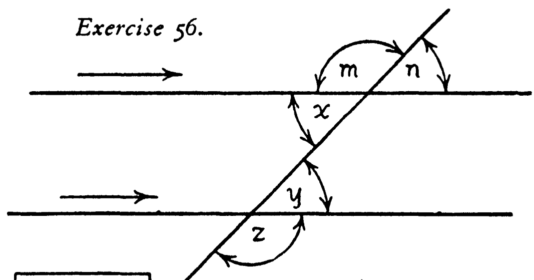


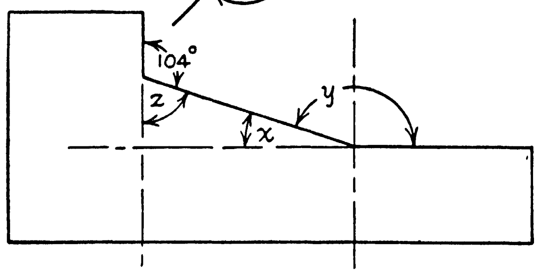
(1) (2)

(1), the projection of AB upon MN is the segment PQ, while the projection of MN upon AB is the segment RS (where AB was first extended in both directions). Similarly in (2), the projection of AB upon MN is PQ; and the projection of MN upon AB is RS.

Exercise 56.

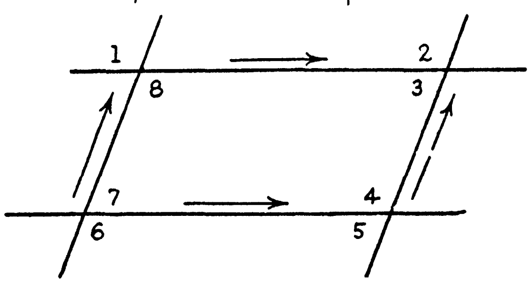
1. (a) If $\angle x = 38°$, find $\angle z$.
   (b) If $\angle m = 112°$, find $\angle y$.
   (c) If $\angle z = 155°$, find $\angle n$.



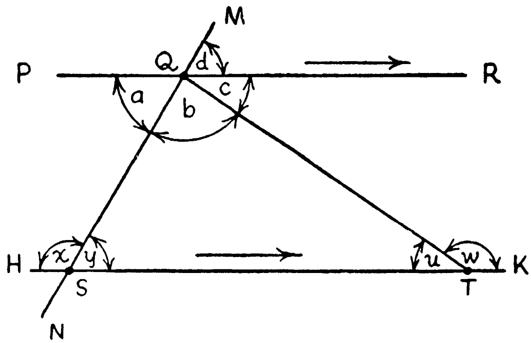


2. In the metal plate shown, find the number of degrees in each of the angles x, y and z.

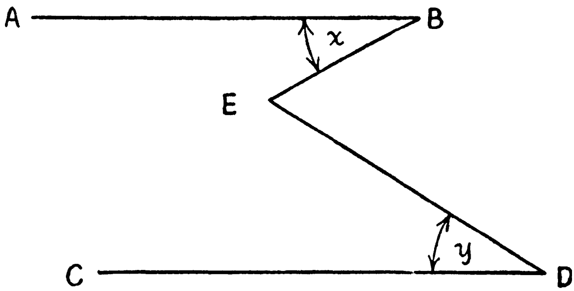
3. (a) If $\angle 1 = 120°$, find $\angle 4$.
   (b) If $\angle 7 = 58°$, find $\angle 3$.
   (c) If $\angle 8 = 115°$, find the sum of $\angle 3 + \angle 4 + \angle 7 + \angle 8$.



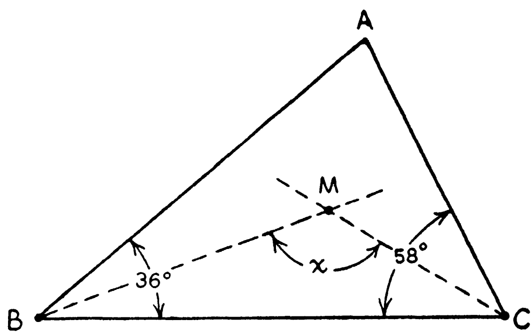
4. (a) If $\angle y = 50°$, and $\angle u = 65°$, find $\angle$MQT.
   (b) If $\angle y = 70°$, and $\angle u = 42°$, find $\angle b$.
   (c) If $\angle b = 85°$, and $\angle u = 45°$, find $\angle x$.
   (d) If $\angle a = 48°$, and $\angle w = 160°$, find $\angle$TQM; also $\angle$PQM.



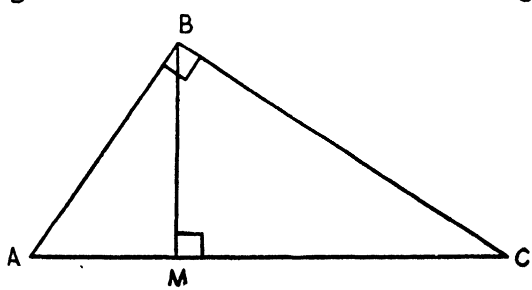
5. Line AB is parallel to CD; if $\angle x = 26°30'$, and $\angle y = 38°15'$, how many degrees are there in $\angle$ BED? (*Hint:* Draw a line through E, parallel to AB.)



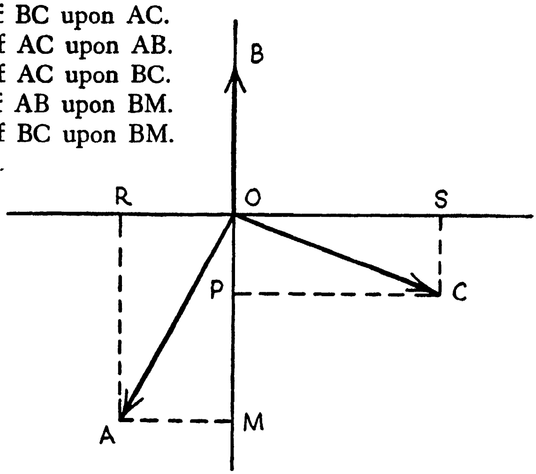
6. If two angles of a triangle are 58° and 36°, respectively, what is the angle (x) formed by their bisectors?



7. If $\angle$ABC is a right angle, and $\angle$AMB is also 90°, name the following:
   (a) the projection of AB upon AC.



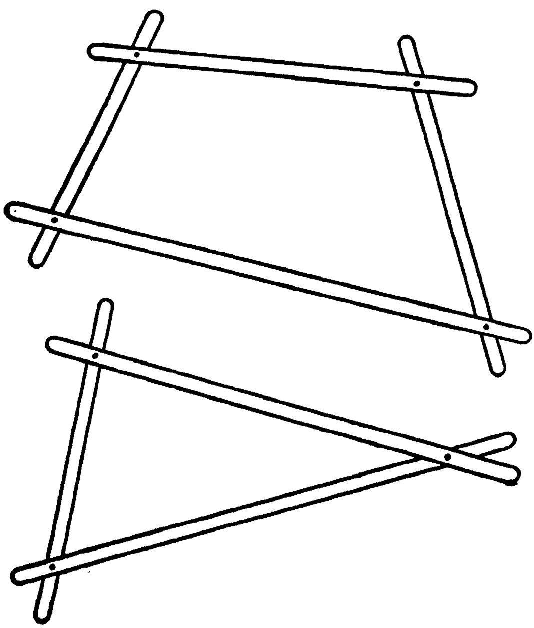
(b) the projection of BC upon AC.
(c) the projection of AC upon AB.
(d) the projection of AC upon BC.
(e) the projection of AB upon BM.
(f) the projection of BC upon BM.

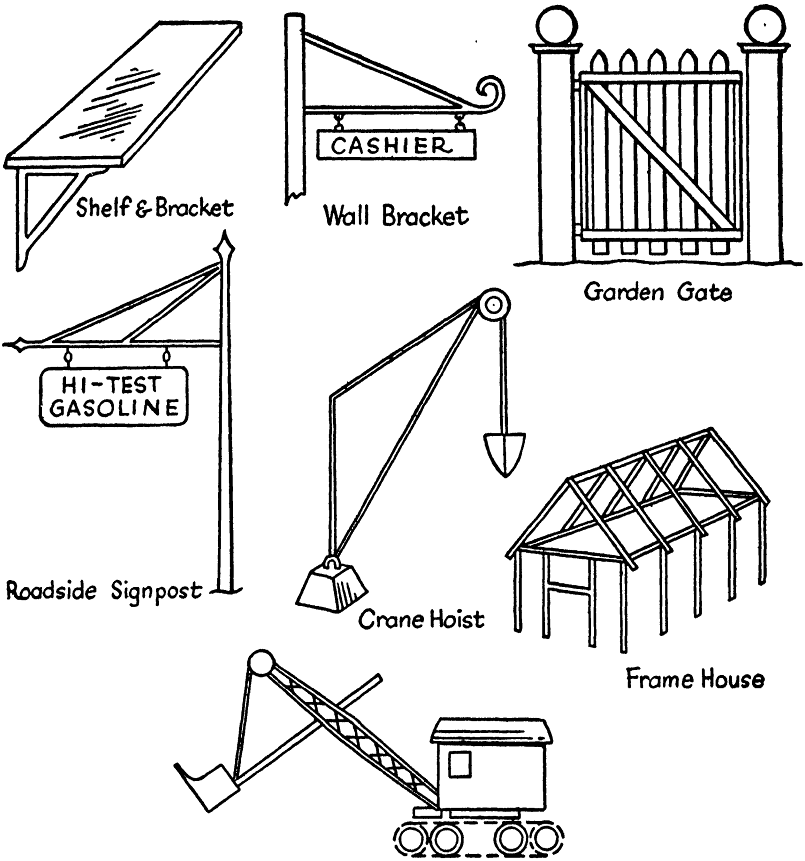


8. In this diagram of forces, state (a) the projection of OA, OB and OC, respectively, upon the horizontal axis; (b) the projections of OA, OB and OC, respectively, upon the vertical axis.

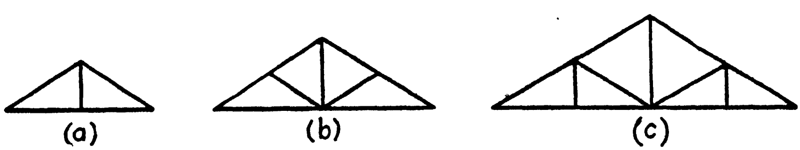
14. TRIANGLES AND POLYGONS

Rigid Frames. If four strips of wood are pinned together with nails, everyone knows that such a frame can easily be deformed, i.e., changed in shape. But if only *three* sticks be similarly fastened together, the resulting triangular frame cannot be changed in shape. In other words, triangles, or three-sided figures, are rigid frames; their *shape* cannot be altered if the lengths of the sides are fixed. For this reason triangular frames are used in structures to secure greater strength and rigidity. In fact, frame structures and trusses are commonly used in engi-

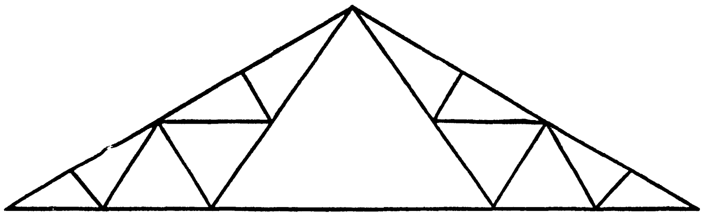




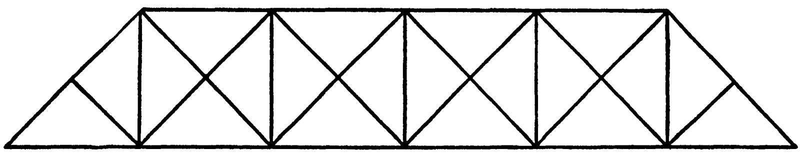
neering and structural work, such as bridges, roofs, pillars, towers, and supporting members and frameworks. A few types of trusses are shown below.



Simple Trusses

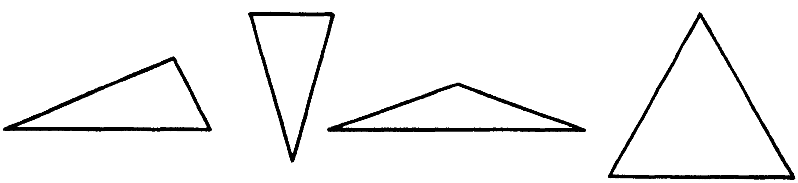


Roof Truss



Railroad Bridge

Kinds of Triangles. A triangle, then, is a straight-line figure having three sides and three angles. The vertices of the angles are also called the vertices of the triangle. If the lengths of the sides have been chosen, the "shape," or angles of the triangle, are automatically determined (fixed). Triangles may be classified according to the lengths of their sides, as follows:



(1) Scalene Triangle (2) Isosceles Triangles (3) Equilateral triangle

In an *equilateral* triangle, all three sides are equal in length.

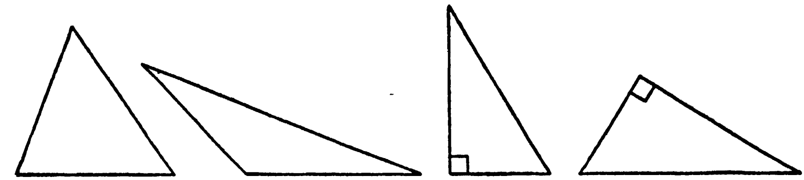
In an *isosceles* triangle, only two sides are equal in length.

In a *scalene* triangle, no two sides are equal in length. In an isosceles triangle, the two equal sides are called the *arms,* and the third remaining side is called the *base.* An equilateral triangle is obviously also isosceles; any one of its sides may be regarded as the base.

Triangles are also classified according to their angles, as follows:

In an *acute* triangle, each of the three angles is less than 90°.

In an *obtuse* triangle, one of the three angles is obtuse, and the two



(1) Acute Triangle (2) Obtuse Triangle (3) Right Triangles

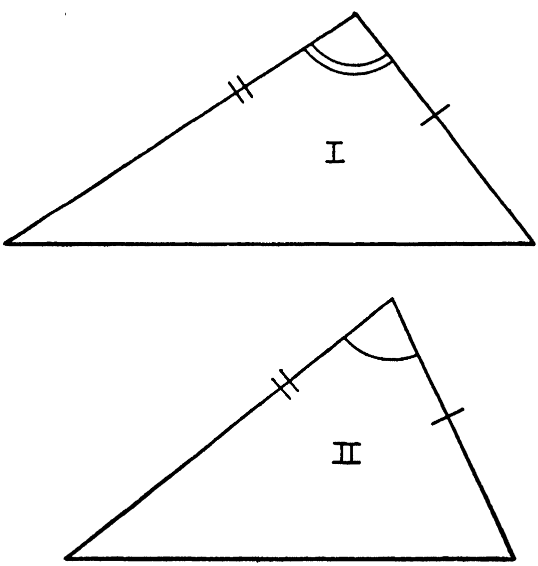
remaining angles are both acute. A triangle cannot have more than one obtuse angle.

In a *right* triangle, one of the angles is a right angle, and the two remaining angles are acute angles. A triangle cannot have more than one right angle.

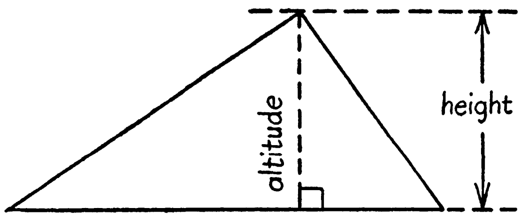
The sum of any two sides of a triangle is always greater than the third side, since a straight line is the shortest distance between any two given points.

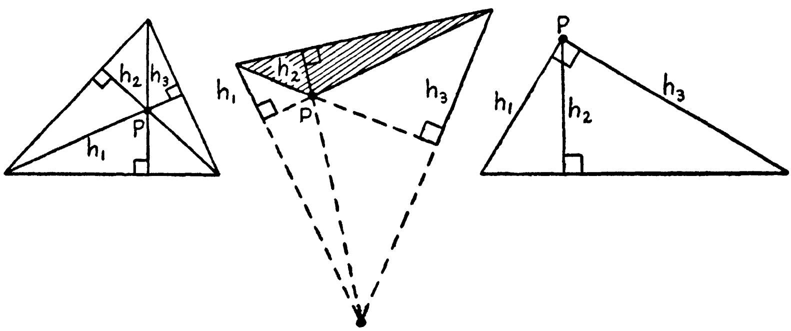
If the angles of a triangle are unequal, the longest side is opposite the largest angle, and *vice versa;* also the shortest side is opposite the smallest angle, and *vice versa.* In a right triangle, the side opposite the right angle is thus the longest side; it is known as the *hypotenuse.*

If two sides of one triangle are respectively equal to two sides of another, but the included angle of the first is greater than the included angle of the second, then the third side of the first triangle is longer than the third side of the second triangle. Conversely, if two pairs of sides are respectively equal, but the third side of one triangle is longer than the third side of a second triangle, then the angle opposite the longer side is greater than the angle opposite the shorter side.



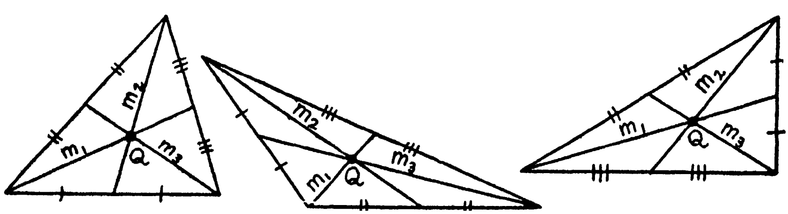
Altitudes and Medians. An *altitude* of a triangle is the perpendicular distance from any vertex to the opposite side. Every triangle therefore has three altitudes, one from each of the three vertices. In an acute triangle, all three altitudes fall *inside* the triangle; in an obtuse triangle, two of the altitudes fall *outside* the triangle; and in a right triangle, two of the altitudes coincide with the two sides.



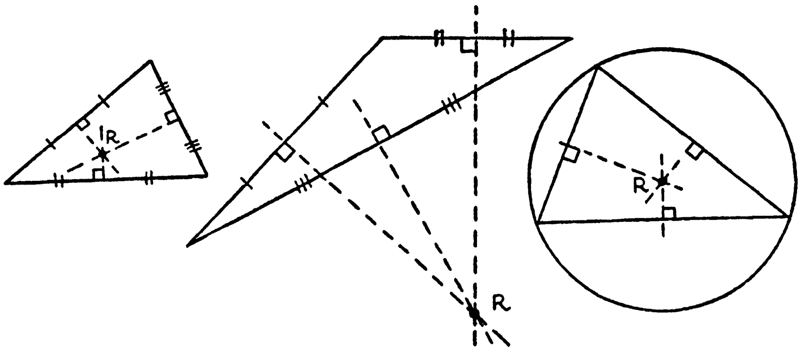


All three altitudes of a triangle (prolonged, if necessary) meet in the same point; this point of intersection (P) lies within, without, or on the triangle, according as the triangle is an acute, an obtuse, or a right triangle, respectively.

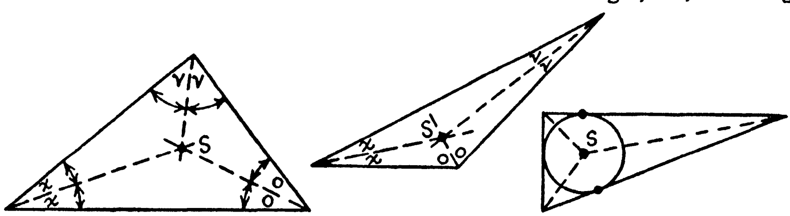
A *median* of a triangle is a line drawn from any vertex to the midpoint of the opposite side. Every triangle has three medians, which must of necessity lie entirely within the triangle. All three medians of a triangle meet in the same point. This point of intersection (Q) divides each median into two segments which are in the ratio of 2:1, respectively; i.e., the shorter segment of the median is, in each case, ⅓ the length of that entire median.

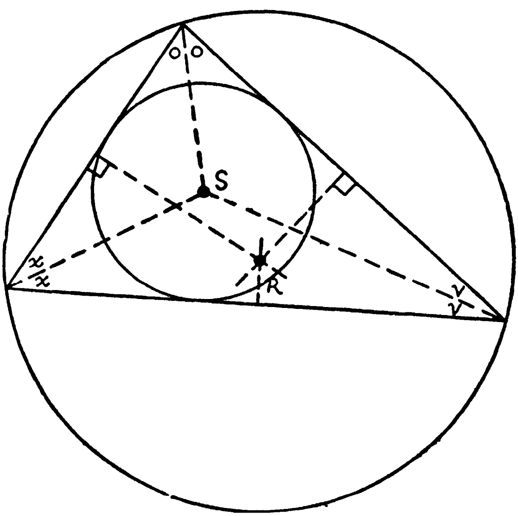


Bisectors of the Sides and Angles. The perpendicular bisectors of the sides of a triangle also meet in a single point. This point is equally distant from the three vertices; hence if it is used as a center, a circumscribed circle can be drawn which will pass through all three vertices.



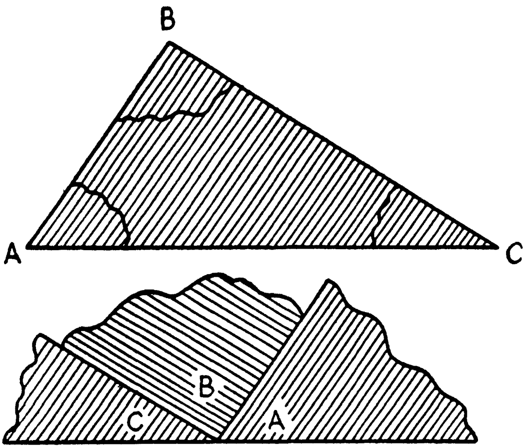
Similarly, the bisectors of the angles of a triangle also meet in a point, but this point is equally distant from all three sides of the triangle. Hence it is the center of the circle *inscribed* in the triangle, i.e., touching

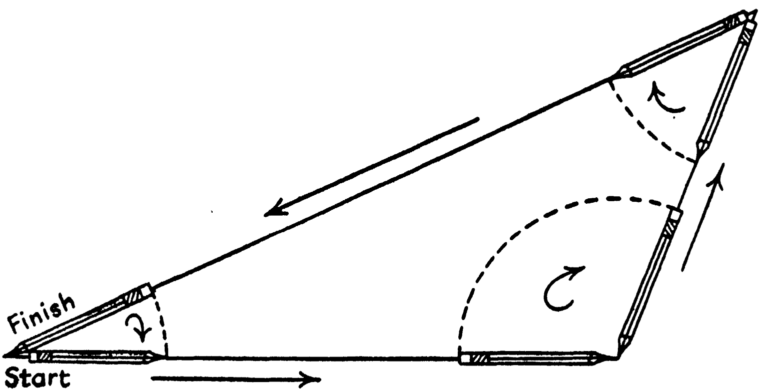




each of the three sides of the triangle in just one point. Only in one exceptional case, as we shall see later, are the centers of the inscribed and circumscribed circles the same, viz., in the case of an equilateral triangle; the equilateral triangle is a "regular polygon," and a circle may be both inscribed within and circumscribed about any regular polygon, both circles having a common center.

Sum of the Angles of a Triangle. One of the most impoıtant properties of a triangle is the fact that, irrespective of its shape, the sum of the three angles always equals 180°. This may be seen by tearing off the three "corners" of a triangular piece of cardboard and fitting the pieces together as shown; when rearranged, the three angles taken together form a straight angle, or 180°. Another way of showing that the angle sum equals 180° is by tracing the sides of a triangle, by sliding and rotating a pencil as suggested in the accompanying sketch. When the triangle has been completely traversed, the pencil will be found to have been reversed in direction; in other words, it has been turned through half a





revolution, or 180°, again showing that the sum of the three angles of the triangle equals a straight angle.

Many properties of geometric figures depend upon this fact. Some of these are:

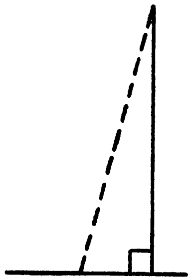
(1) That no triangle may have more than one right angle or one obtuse angle.

(2) That only one perpendicular may be drawn to a line from a point outside.

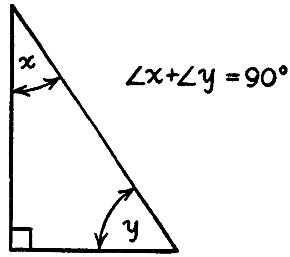
(3) That the acute angles of a right triangle are complementary.

(4) That an exterior angle of a triangle (formed by prolonging any side) is equal to the sum of the two remote interior angles.

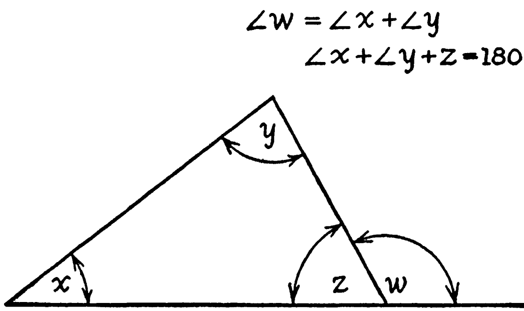
(5) That angles included between parallels, and on the same side of a transversal, are supplementary.



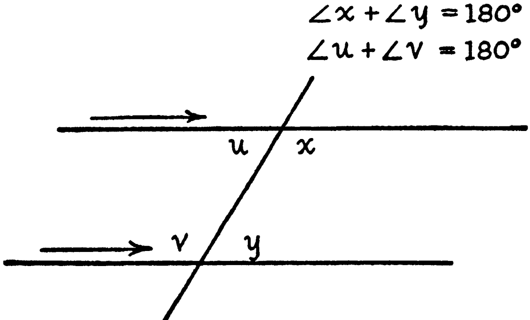
(2)



(3)



(4)

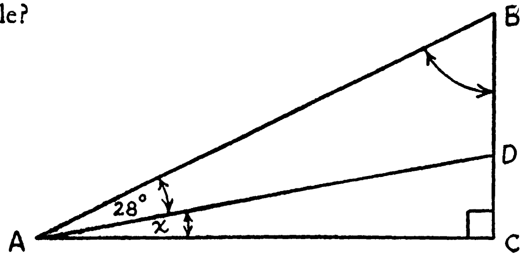


(5)

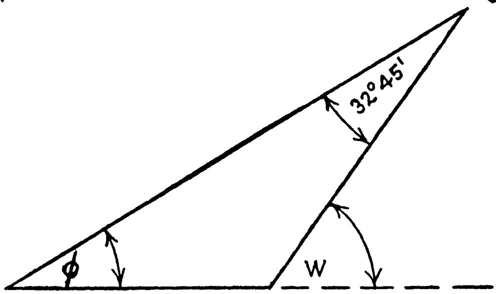
Exercise 57.

1. If one acute angle of a right triangle equals 14°36′, what is the value of the other acute angle?

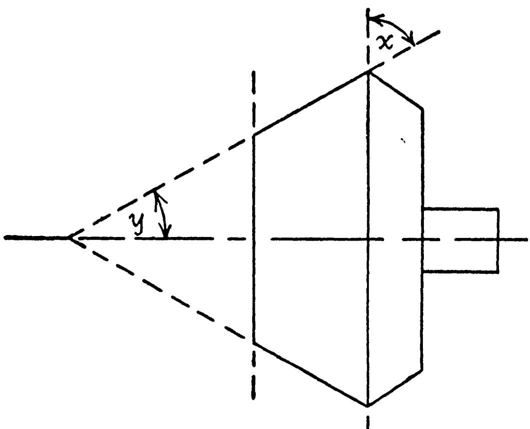
2. In the right triangle ABC, find ∠ x if ∠ B equals 46°.



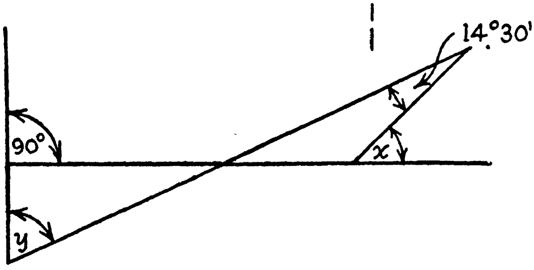
3. If ∠ w=78°12′, find the value of ∠ ϕ.



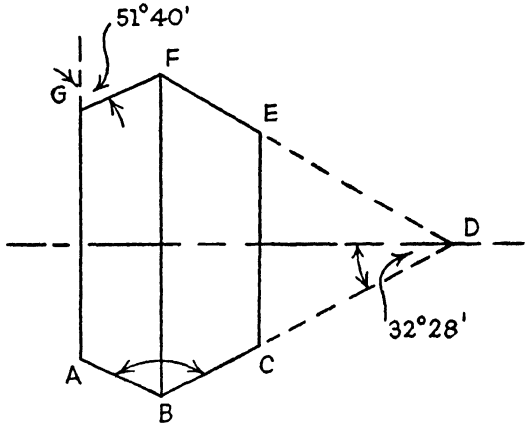
4. In this sketch of a bevel gear blank, if ∠ y=61°24′ 10″, find ∠ x.



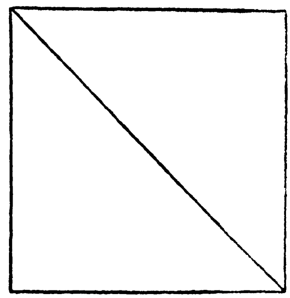
5. Find the value of ∠ x, if ∠ y equals 39°52′.



6. Find $\angle ABC$, if the face angle at G is $51°40'$ and the half-angle at D equals $32°28'$.

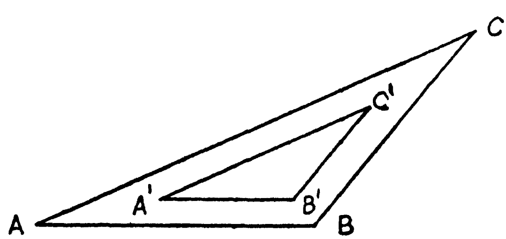


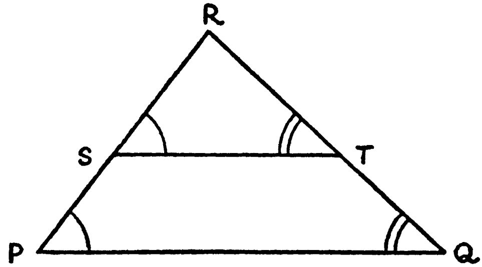
Congruence, Similarity, and Equivalence. If two triangles can be made to coincide or "fit exactly," they are said to be *congruent*. Congruent triangles might be said to have the same "shape" and the same "size." Any two triangles are congruent if:



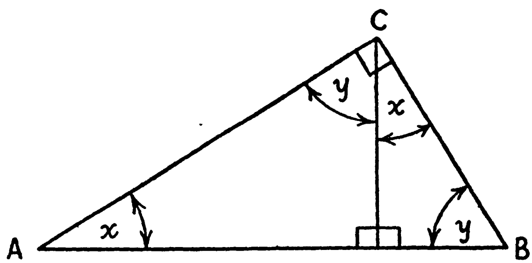
(1) two sides and the included angle of one triangle are equal respectively to two sides and the included angle of the other;
(2) two angles and the included side of one triangle are equal respectively to two angles and the included side of the other;
(3) if a side and any two angles of one triangle are equal respectively to a side and the corresponding angles of the other;
(4) if all three sides of one triangle are respectively equal to the three sides of the other (even though nothing is known about the angles);
(5) if they are right triangles, and the hypotenuse and one side of one triangle are equal respectively to the hypotenuse and corresponding side of the other.

Two triangles are not necessarily congruent, however, merely because the three angles of the one are equal respectively to the three angles of the other; in this case they are said to be *similar*



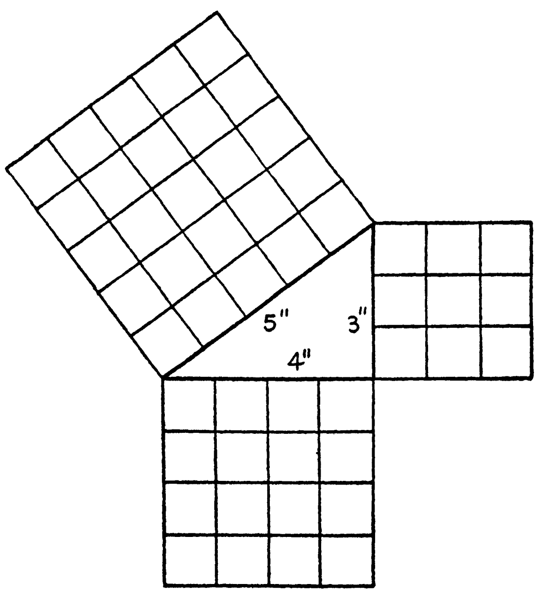


triangles. Such triangles have the same "shape," but not the same "size." We shall learn more about similar triangles later.



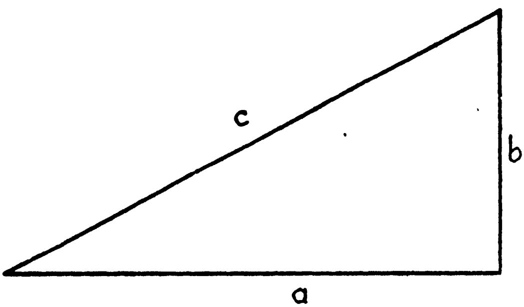
Triangles which are neither congruent nor similar may nevertheless be equivalent in area (i.e., cover the same surface), just as, for example, rectangle 4″ ×9″ covers the same surface as a square 6″ ×6″; the rectangle and the square are not congruent, neither are they similar in shape, yet they are *equivalent* in area. We shall learn more about this, too, later on.

The Right Triangle Rule. One of the most famous and useful geometric relations is the relation between the lengths of the three sides of a right



Since $3^2+4^2=5^2$, or $9+16=25$, it is clear that if we *square* the number of units in each side and add these results, the sum obtained will be the square of the number of units in the hypotenuse.

triangle, viz., that $a^2+b^2=c^2$. Put into words: the square of the hypotenuse equals the sum of the squares of the other two sides. By means of this relation we can find the length of any side of a right triangle if we know the lengths of the other two sides. We simply use one of the following formulas:



$c^2=a^2+b^2$ or $c=\sqrt{a^2+b^2}$
$a^2=c^2-b^2$ or $a=\sqrt{c^2-b^2}$
$b^2=c^2-a^2$ or $b=\sqrt{c^2-a^2}$

Rule 1. To find the hypotenuse of a right triangle, find the square root of the sum of the squares of the other two sides.

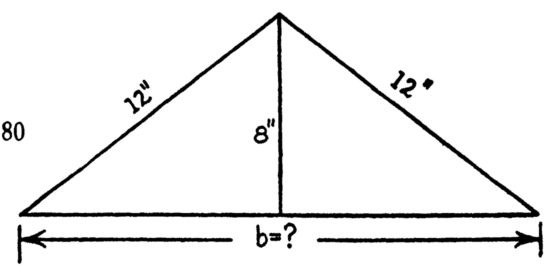
Rule 2. To find either side of a right triangle, if the other side and the hypotenuse are known, find the square root of the difference between the square of the hypotenuse and the square of the known side.

EXAMPLE 1: Find the length of the base of the isosceles triangle shown.

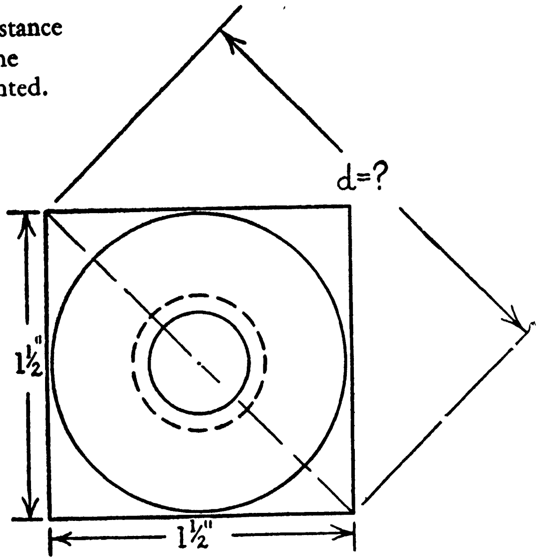
SOLUTION: The altitude, 8″, divides the base in half. Call the half-base x.

$x^2+8^2=12^2$
$x^2=12^2-8^2=144-64=80$
$x^2=80$
$x=\sqrt{80}=8.94+$,

Therefore the base= $2\times8.94=17.9''$, *Ans.*



Example 2: Find the distance across the corners of the squarehead nut represented.



Solution:

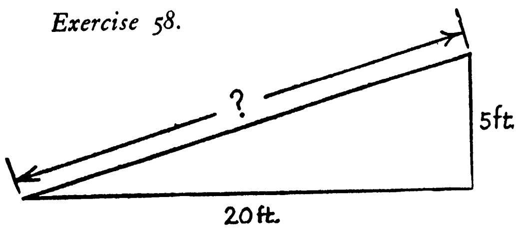
$d^2=(3/2)^2+(3/2)^2$

$d^2=9/4+9/4=18/4=4.5$

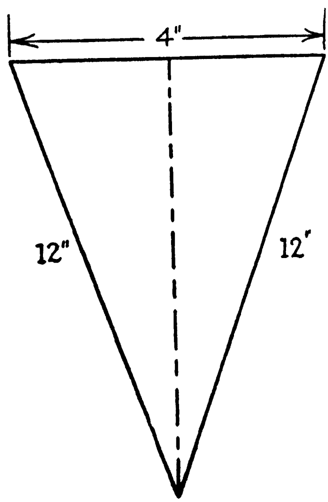
$d=\sqrt{4.5}=2.1''$, *Ans.*

Exercise 58.

1. If the base of an isosceles triangle is 16″ and the altitude is 9″, find the length of the equal sides.



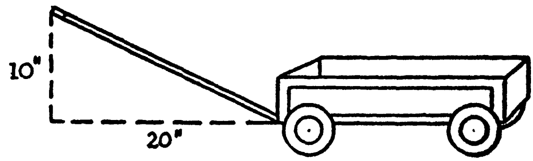
2. The runway of a garage ramp rises 5 ft. in a horizontal distance of 20 ft. Find the length of the ramp.



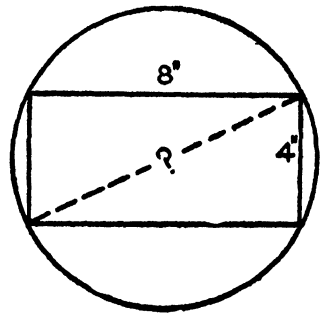
3. A wedge in the shape of an isosceles triangle has the dimensions shown. Find the length of the center line.

4. If the normal lens used in a camera has a focal length equal to the diagonal of the film which the camera accommodates, what should be the focal length of the lens for a camera using 4″×5″ plates?

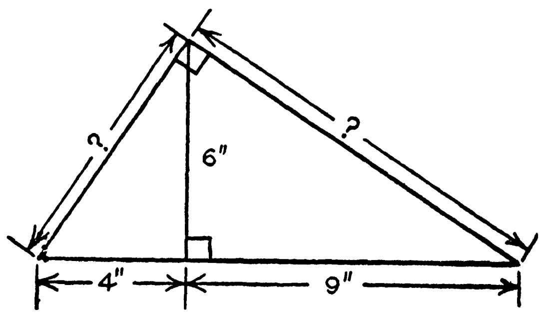
5. A child pulling a cart stands 20″ in front of it. One end of the handle is 10″ higher than the other. How long is the handle?



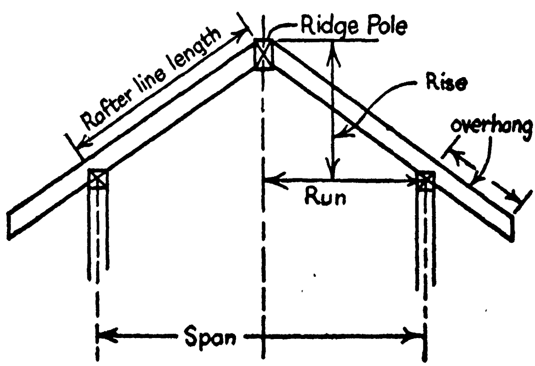
6. A wooden beam 4″×8″ is to be cut from a log. What is the diameter of the smallest log that could be used?



7. The altitude of a right triangle drawn to the hypotenuse is 6″. If the altitude divides the hypotenuse into segments of 4″ and 9″, find the lengths of the other two sides. How could you check your result?

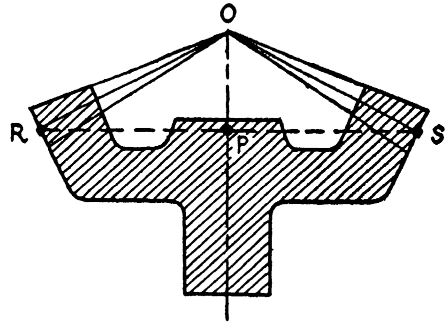


8. With an overhang of 18 in., a rise of 16 ft. and a span of 28 ft., find the overall length of rafter required.

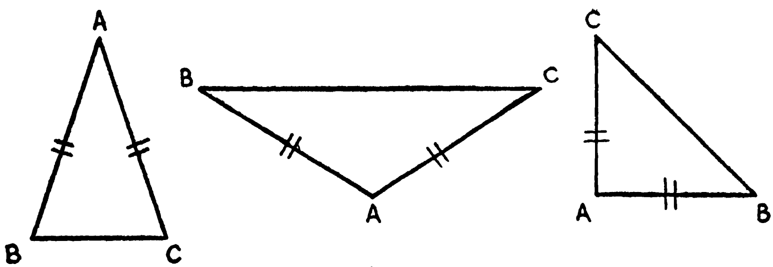


9. A square is inscribed in a circle whose radius is 4″. Find the length of the side of the square.

10. The distance OR in a bevel gear is called the apex distance, and RS is called the pitch diameter. If OP is 2½″ and OR is 5½″, find the pitch diameter.



Isosceles Triangle. Any triangle having at least two of its sides of equal length is called an *isosceles* triangle. The third side (BC) is referred to as the *base,* although it need not necessarily be the side on which the



Isosceles Triangle

triangle is "standing." That vertex (A) of the triangle which lies opposite the base is called *the vertex* of the isosceles triangle. The altitude drawn from the vertex to the base has the following properties:

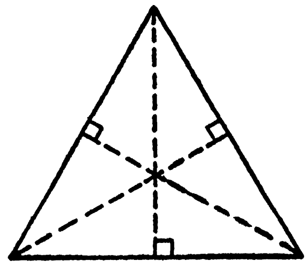
(1) It bisects the base (it is thus a median).
(2) It bisects the angle at the vertex (the "vertex angle").
(3) It divides the isosceles triangle into two congruent triangles.

Another property of the isosceles triangle is that its *base angles* are equal; these are the two angles opposite the equal sides, i.e., adjacent to the base. It is also true that if any two angles of a triangle are equal, then the sides opposite them are equal, i.e., the triangle is isosceles.

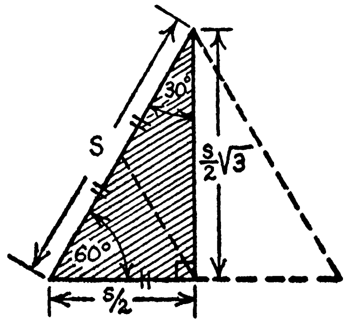
Special Triangles. An *equilateral triangle* has special properties, not found in other triangles. Thus its three altitudes are all equal in length;

so are the three medians. In fact, the medians and altitudes are identical with one another; they are also identical with the angle-bisectors and with the perpendicular bisectors of the sides. A general formula for the altitude of an equilateral triangle is

$$h=\frac{s}{2}\sqrt{3}, \text{ where } s=\text{length of a side.}$$

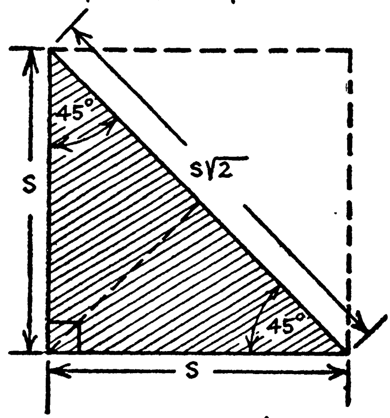


Another "special" triangle is the so-called *30°—60°—90° triangle,* or half an equilateral triangle. It is clear that in such a triangle, the hypotenuse is twice as long as the side opposite the 30° angle, or the shortest side. Moreover, the median drawn from the vertex of the right angle to the hypotenuse equals half the hypotenuse in length.



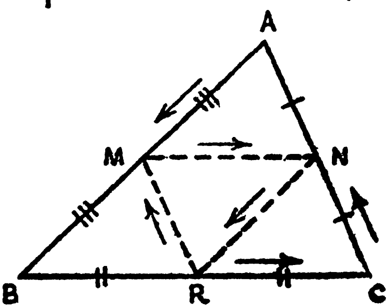
The *isosceles right triangle* is another special triangle often encountered. It is simply half of a square. In an isosceles right triangle, the altitude drawn to the hypotenuse equals

$$\frac{s}{2}\sqrt{2}.$$

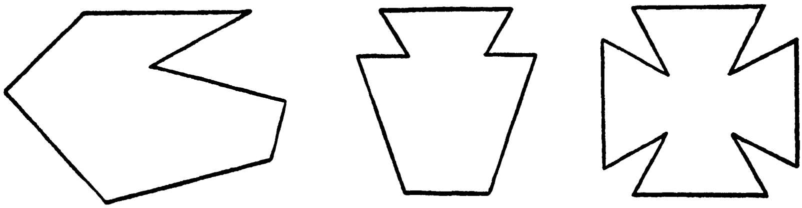


Mid-join of a Triangle. A line joining the midpoint of any side of a triangle with the midpoint of any other side is sometimes called the *mid-join* of the triangle. The mid-join has two important properties: (1) it is parallel to the third side of the triangle; (2) it is equal in length to *half* the third side.

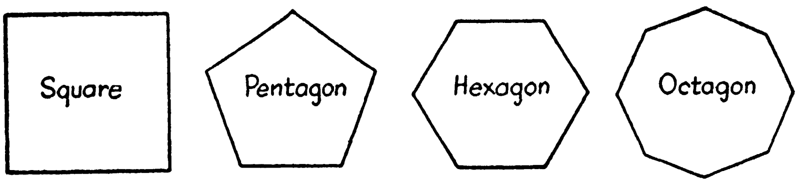
Thus if BM=MA and AN=NC, then MN (the mid-join) is parallel to BC and equal to ½ (BC). If the other two mid-joins be drawn, the original triangle is divided into four congruent triangles and three equivalent parallelograms.



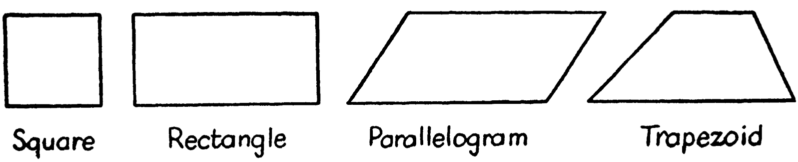
Polygons. Any rectilinear figure having four or more sides is called a *polygon*. Polygons may be irregular or regular. Irregular polygons may



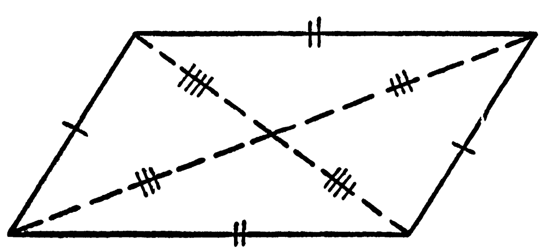
be *convex* or *reëntrant*. A *regular* polygon is a convex polygon all of whose sides are equal and all of whose angles are equal. Examples of regular polygons are shown below; both kinds of polygons are frequently encountered in shop practice and trade operations.



Quadrilaterals. Four-sided polygons are called *quadrilaterals*. Such figures are not rigid, as in the case of the triangle. In other words, the shape of a quadrilateral is *not* determined by the lengths of its sides; a figure of four fixed-length sides may be distorted by changing its angles *without* changing the lengths of its sides. The sum of the four angles of every quadrilateral, however, is constant, and equals 360°. Important types of quadrilaterals are shown below.

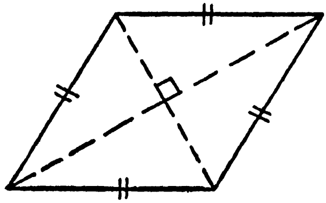


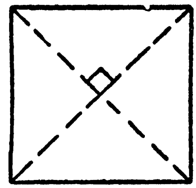
Parallelograms. A quadrilateral having its opposite sides parallel in pairs is called a *parallelogram*. Some of the most important properties of all parallelograms are:



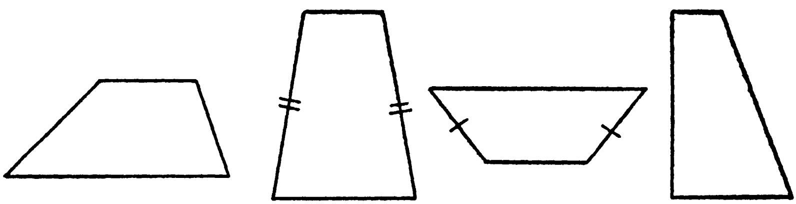
(1) the opposite sides are equal.
(2) the opposite angles are equal.
(3) the diagonals bisect each other.

The *square* and the *rectangle* are special cases of the parallelogram; in both cases, all four angles are equal, each being 90°; the square differs from the rectangle in being equilateral, i.e., all four sides are equal. A parallelogram (other than a square) which is equilateral is called a *rhombus*. A special feature of the rhombus, which does not hold for non-equilateral parallelograms, is the fact that its diagonals not only bisect each other, but intersect at right angles as well, a feature that holds for squares also. The *altitude* of any parallelogram is the perpendicular distance between a pair of bases; either pair of parallel sides may be regarded as bases.





The Trapezoid. Any quadrilateral having two and only two of its sides parallel is known as a *trapezoid;* if the remaining two sides are equal in length it is an *isosceles trapezoid.* In either case, the two parallel sides are called the *bases* of the trapezoid. Trapezoidal shapes are frequently encountered in various machine parts and in construction work. The midjoin of a trapezoid, also commonly called the *median* of the trapezoid, is a line joining the mid-points of the two non-parallel sides. The median of a trapezoid is



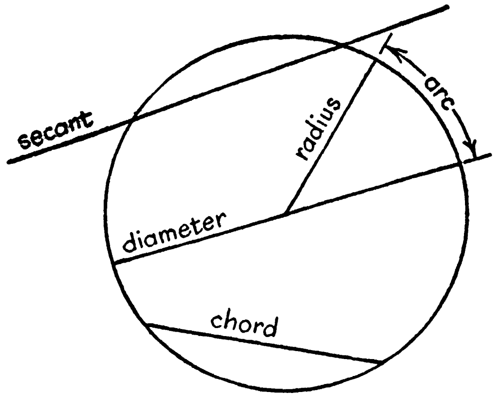
(1) parallel to the two bases; and
(2) equal to half the sum of the two bases.

The *altitude* of a trapezoid is the perpendicular distance between the two bases.

15. CIRCLES AND TANGENTS

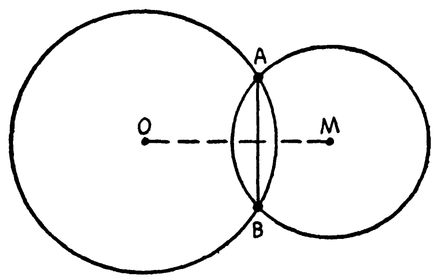
Circles. The circle is one of the earliest geometric forms known to primitive man. It was doubtless suggested by the rims of his pottery or the edge of spherical fruit when cut in half.

A *circle* is simply a closed line, every point of which is equally distant from a point within called the *center.* The distance from the center to the circle, or a line from the center to any point on the circle, is called the *radius.* Obviously, all the radii of any given circle are equal to one another. A straight line can intersect a circle in two points and only two; such a line is called a *secant.* If the ends of a line terminate in a circle, the line is called a *chord.* A *diameter* is thus also a chord; it is the longest chord that can be drawn in a circle, and is exactly twice the length of a radius. Naturally, all diameters of the same circle are equal.

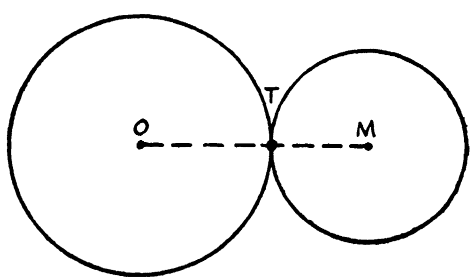


An *arc* of a circle is that part of the circle included between any two particular points. The ends of a diameter divide the circle into two equal arcs. If an arc is less than half the circle, or *semicircle,* it is a *minor* arc; if greater than a semicircle, it is a *major* arc. The straight line joining the ends of an arc is called the *chord* of that arc; a chord is said to subtend the arc. A diameter is thus a chord subtending a semicircle.

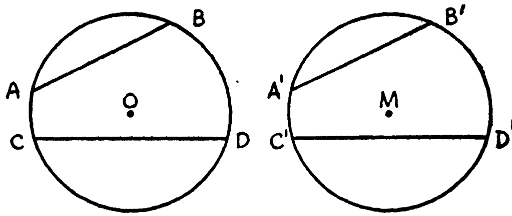
Two circles can intersect each other in only two points; the chord joining these points is called a common chord. Circles may also be *tangent* to each other, which means that they touch each other in one point and only one point; this common point is called the *point of tangency.* The line joining their centers, or line of centers, passes through the point of tangency. In the case of intersecting circles, the line of centers is the perpendicular bisector of the common chord.

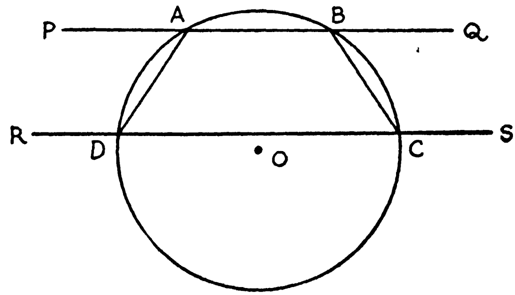


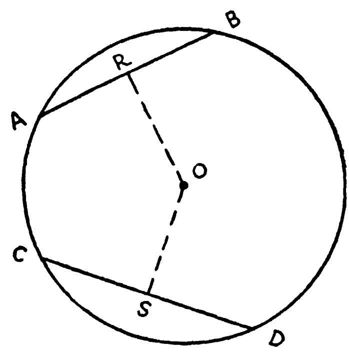
Chords. It is readily seen that in the same circle (or in two equal circles), if two chords are equal in length, then their arcs are equal; conversely, if two arcs are equal in length, then their subtended chords are equal. Furthermore, if in the same circle (or in two equal



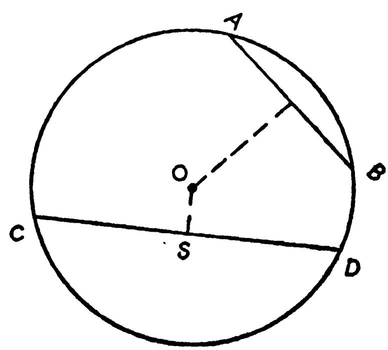
circles) two chords are equal in length, they must be equally distant from the center of the circle; and conversely, if two chords are equally distant from the center, then the chords are equal. If, on the other hand, two chords are *unequal,* then their distances from the center are unequal, and the greater chord is at the smaller distance. In other words, the nearer a chord approaches the center, the longer it becomes, and vice versa. Conversely, if two chords are at unequal distances from the center, the chords are unequal, and the one farthest from the center is the shorter chord. Any line passing through the center of a circle and perpendicular to a chord will bisect both the chord and its subtended arc; conversely, any perpendicular bisector of a chord must pass through the center of the circle.





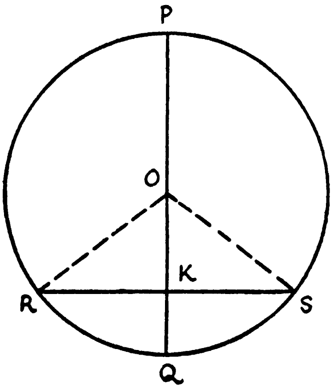


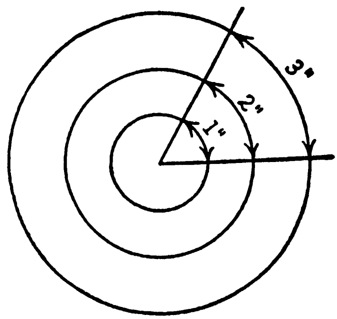
Circles and Angles. Any angle formed by two radii is called a *central angle;* its vertex is, of course, at the center. A central angle is said to be measured by its *intercepted arc;* this means that the number of degrees in the arc is the same as the number of degrees in its central angle, and conversely. An arc, in other words, may be described as being of so-and-so many degrees, or as so many inches, etc., long. How to measure an arc in *linear* units instead of *angular* units will be described later; here it should be



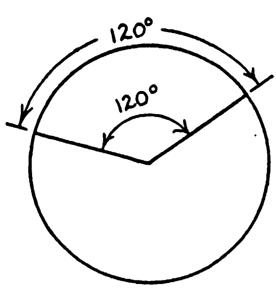
noted that in a series of *concentric circles,* or circles having the same center but different radii, a central angle of a fixed number of degrees will cut off arcs of different *absolute* lengths, even though all of these arcs are of the same *relative* length, i.e., contain the same number of angular units, or degrees. Or, putting it another way, the *actual* length of an arc depends not only upon the number of degrees of arc, but also on the radius of the circle.

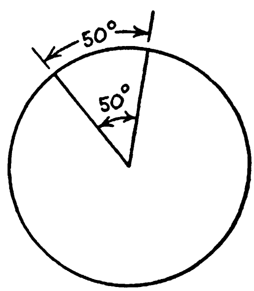
In any given circle, therefore, or in any two equal circles, equal central

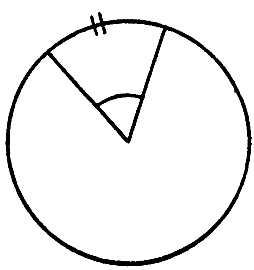


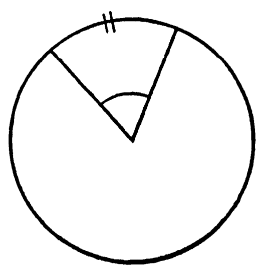


angles intercept equal arcs, and equal arcs are intercepted by equal central angles. Furthermore, if two central angles are unequal, their intercepted arcs are unequal, and the greater angle intercepts the greater arc; conversely, if two arcs are unequal, their central angles are unequal, and the greater arc is intercepted by the greater angle.

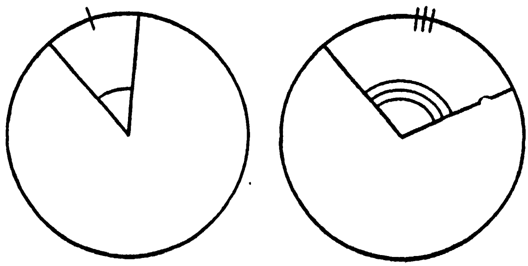




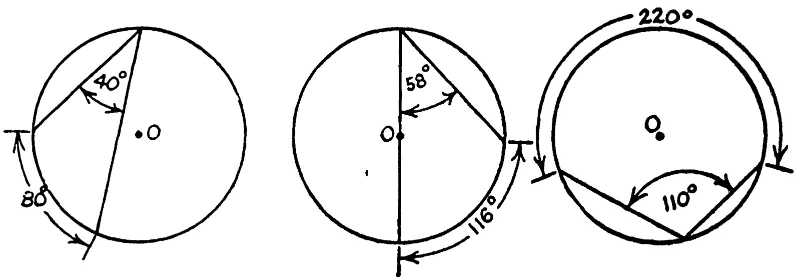


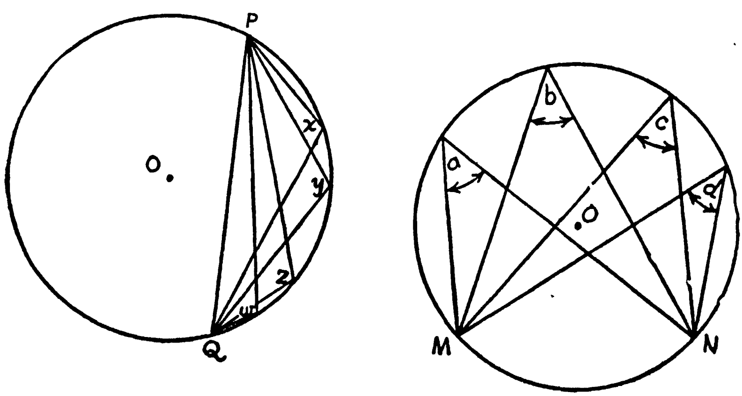


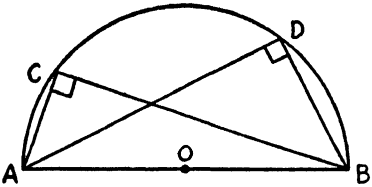
An inscribed angle of a circle refers to

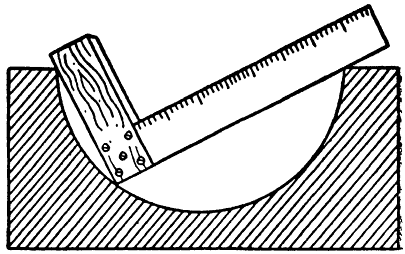


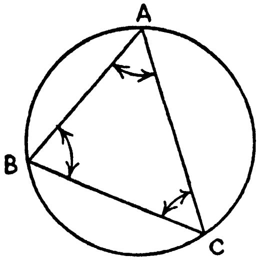
any angle formed by two chords (or a chord and a diameter) which intersect on a point on the circle. An inscribed angle is always measured by *half* its intercepted arc; or, the arc contains *twice* as many degrees as the inscribed angle which intercepts it. From this it will be seen that *all* angles inscribed in the same *segment* are equal, a segment being a figure bounded by an arc and its subtended chord. Thus in the figure, $\angle x = \angle y = \angle z = \angle w$; also, $\angle a = \angle b = \angle c = \angle d$, etc. As a special case of this, we note that all angles inscribed in a semicircle must be right angles. This is the relation a carpenter uses to test the accuracy of a semicircular groove; if the vertex of the carpenter's square touches every point of the groove as the square slides around, the groove is a true semicircle.

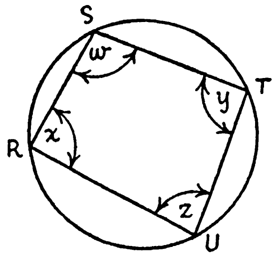


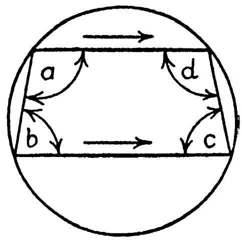












Other interesting and significant relations also become clear.

(1) Verification of the sum of the angles of a triangle;

$$\angle A = \tfrac{1}{2}\,\overset{\frown}{BC}$$
$$\angle B = \tfrac{1}{2}\,\overset{\frown}{AC}$$
$$\angle C = \tfrac{1}{2}\,\overset{\frown}{AC}$$

$$\angle A + \angle B + \angle C = \tfrac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{AC} + \overset{\frown}{AC}) = \tfrac{1}{2}(360^\circ) = 180^\circ.$$

(2) The opposite angles of an inscribed quadrilateral are supplementary;

$$\angle w = \tfrac{1}{2}\,\overset{\frown}{RUT}$$
$$\angle z = \tfrac{1}{2}\,\overset{\frown}{RST}$$

$$\angle w + \angle z = \tfrac{1}{2}\overset{\frown}{RUT} + \tfrac{1}{2}\overset{\frown}{RST} = \tfrac{1}{2}(\overset{\frown}{RUT} + \overset{\frown}{RST}) = 180^\circ.$$

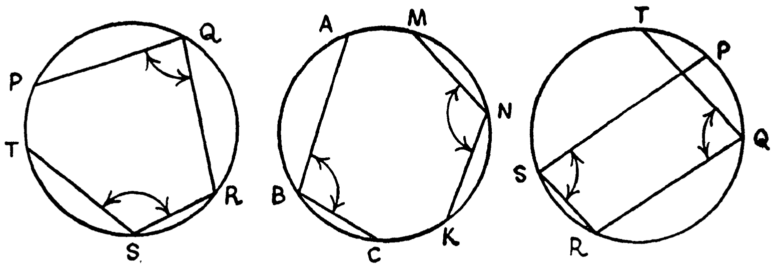
Similarly, $\angle x + \angle y = \tfrac{1}{2}\overset{\frown}{STU} + \tfrac{1}{2}\overset{\frown}{SRU} = \tfrac{1}{2}(360^\circ) = 180^\circ.$

(3) A trapezoid inscribed in a circle must be an isosceles trapezoid;

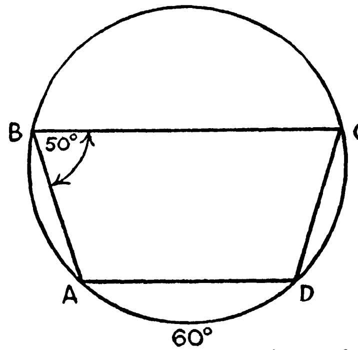
$\angle b + \angle a = 180^\circ$, or $\angle b = 180^\circ - \angle a$
$\angle c + \angle a = 180^\circ$, or $\angle c = 180^\circ - \angle a$
therefore $\angle b = \angle c$.

Exercise 59.

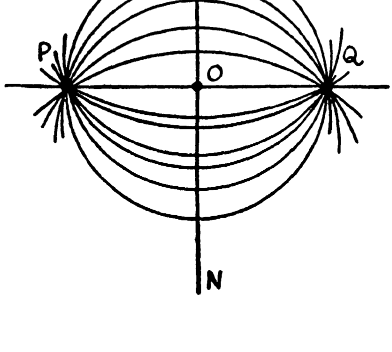
1. **Find the sum of $\angle Q + \angle S$, if arc PT=40°.**
2. **How many degrees are there in $\angle B + \angle N$, if arc AM=30° and arc CK=40°?**

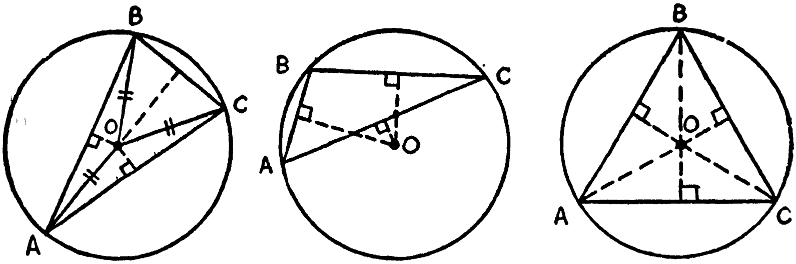


3. Find the sum of $\angle S + \angle Q$, if arc PT=44°.
4. Show why an oblique parallelogram cannot be inscribed in a circle.
5. If ABCD is an inscribed trapezoid, and $\angle ABC = 50°$ and arc AD=60°, find the number of degrees in each of the other arcs.
6. A quadrilateral PQRS is inscribed in a circle, and the two diagonals are drawn; arc PQ=80°, arc QR =150°, and arc RS=40°. Find all the angles in the figure.



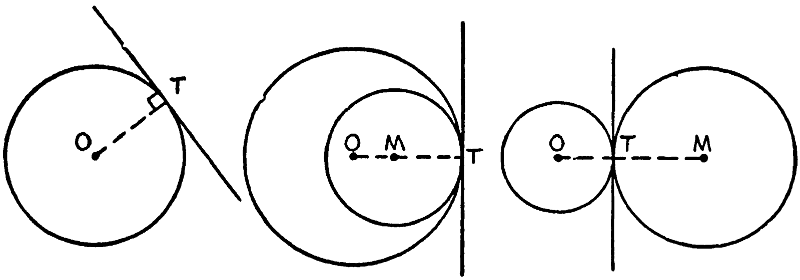
Circumscribed Circles. Geometric figures whose vertices lie on a circle and whose sides are chords of the circle, are said to be *inscribed* in the circle; or, the circle is circumscribed about the figure. A little consideration will show that through any *two* given points it is possible to draw an infinite number of circles, each of which has its center somewhere on the perpendicular bisector of the line PQ joining the two given points. But if it is attempted to draw a circle through any *three* given points, only one circle is possible. Thus, one and only one circle may be circumscribed about a given triangle. As we have already seen, the perpendicular bisectors of the three sides of any triangle all meet in a point which is equidistant from the vertices of the triangle. Hence this common point of intersection is the center of the circumscribed circle; its radius is the distance from this point to any one of the vertices (OA=OB=OC).



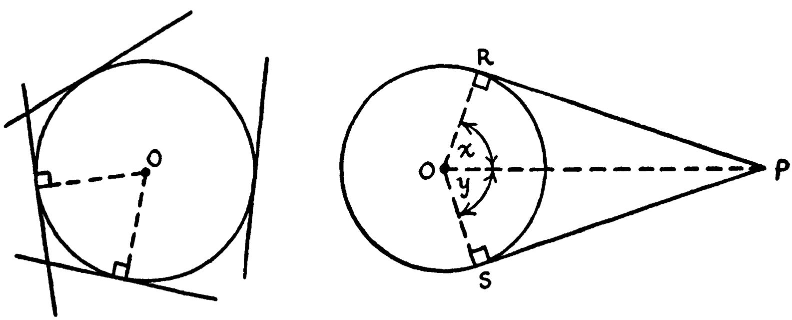


Acute Triangle Obtuse Triangle Equilateral Triangle

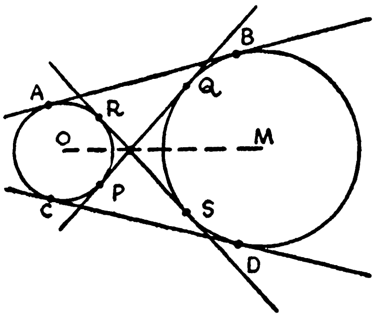
Circles and Tangents. A straight line is said to be *tangent* to a circle if the line touches the circle in one and only one point, no matter how far it is extended. Two circles are said to be tangent if they have but one point in common; they may be *internally* tangent or *externally* tangent. The point of tangency is also called the point of contact. A radius

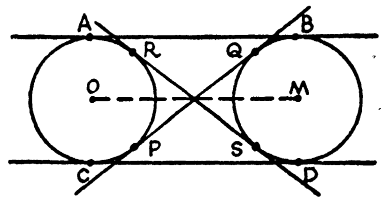


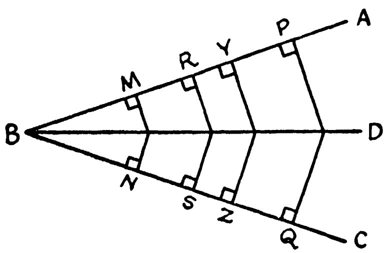
drawn to a tangent at the point of contact is always perpendicular to the tangent; likewise, any line perpendicular to a tangent at the point of contact will pass through the center. In the case of tangent circles, the line of centers passes through the point of tangency.

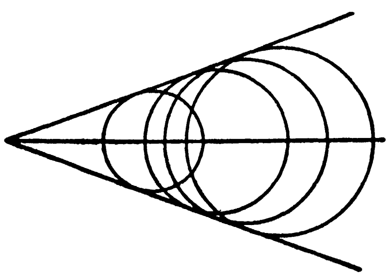


Any number of tangents may be drawn to a given circle, but at different points on the circle; at any *particular* point on a circle *only one* tangent can be drawn, however. From a point outside a circle, exactly two tangents may be drawn to the circle and no more. These tangents are equal, i. e., PR=PS. Moreover, the line from the outside point (P) to the center bisects the angle between the tangents, i.e., ∠RPO=∠SPO; also, ∠*x*=∠*y*. Where two non-tangent, non-intersecting circles are given, two external and two internal tangents may be drawn. These pairs of tangents are respectively equal, i.e., AB=CD, and RS=PQ. If the circles





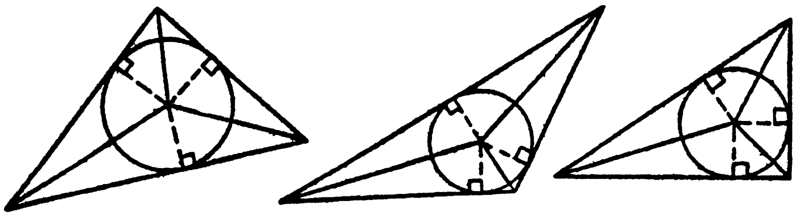




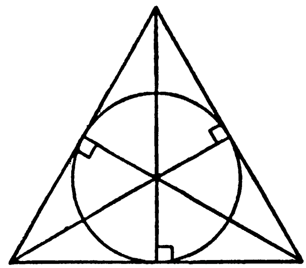
are equal, the external tangents are parallel as well. In both instances the line of centers passes through the point of intersection of the pair of internal tangents.

The Bisector of an Angle. Every point on the bisector of an angle lies at equal distances from the sides of the angle. Thus a series of circles could be "inscribed" within the angle, i.e., the sides of the angle would be tangent to the respective circles. This principle is employed when inscribing a circle in a triangle.

Inscribed Circles. A circle is said to be inscribed in a figure if it is tangent to every side of the figure. If a circle is inscribed in a triangle, the center of the circle is always the common point of intersection of the three angle-bisectors, and its radius equals the perpendicular distance from the center to the respective sides.

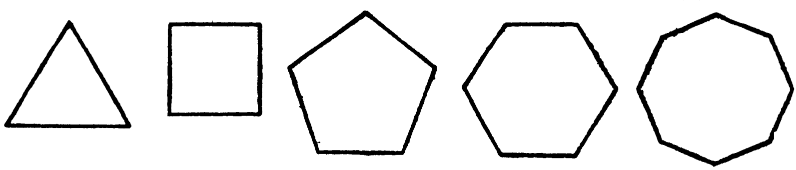


Acute Triangle Obtuse Triangle Right Triangle

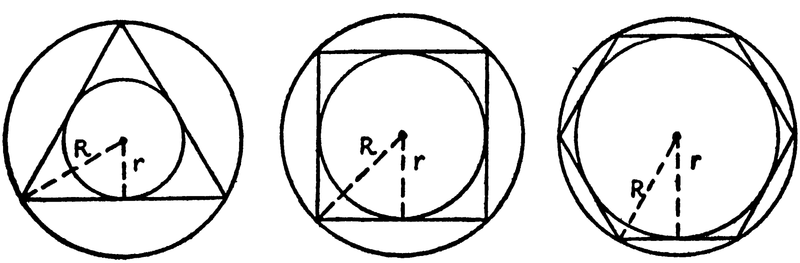


Equilateral Triangle

Regular Polygons. If all the sides and all the angles of a polygon are equal, the figure is said to be a *regular polygon.* A polygon may have equal angles, but unequal sides; or, it may have equal sides, but unequal angles; in neither case, however, is it a regular polygon. A circle may always be both inscribed in, and also circumscribed about, any regular polygon, no matter how many sides it may have; the inscribed and circumscribed circles in each case have the same center. The radius of the circumscribed circle is called the radius of the polygon (R); the radius of the inscribed circle is called the *apothem* of the polygon (r). In the



Equilateral Triangle Square Regular Pentagon Regular Hexagon Regular Octagon



case of a square, R=half the diagonal, and r=half the side. In the case of a regular hexagon, R equals the side of the hexagon. In the case of the equilateral triangle. $R=2r$. In all cases, the central angle of a regular polygon is the angle included between any two consecutive radii; in a regular polygon of n sides, the central angle equals $\left(\frac{360}{n}\right)^{\circ}$.

Circumference of a Circle. The length of a circle is called its *circumference.* In every circle, however large or small, the ratio of the circumference (C) to its diameter (D) is constant, and equal to 3.1416, or approximately $3\frac{1}{7}$ .Thus ratio is called π (pi). Thus any circumference is about $3\frac{1}{7}$ times as long as its diameter; that is

$$\frac{C}{D}=\pi=3.1416, \text{ or } C=\pi D.$$

Since $D=2R$, then we also have

$$C=2\pi R.$$

EXAMPLE 1: Find the circumference of a circle with radius 14″.

SOLUTION:

$$C=2\pi R$$

$$C=2\times\frac{22}{7}\times 14$$

$$=2\times\frac{22}{7}\times 14=88'', \textit{ Ans.}$$

EXAMPLE 2: If the circumference of a circle is 110 ft., find its diameter.

SOLUTION:

$$C=\pi D, \text{ or } D=\frac{C}{\pi}$$

$$D=110\div\frac{22}{7}=110\times\frac{7}{22}=35 \text{ ft., } \textit{Ans.}$$

For ordinary rough computations, the value $\pi=3\frac{1}{7}$ may be used; for more exact work, use $\pi=3.14$; and for very accurate work, use $\pi=3.1416$.

Length of an Arc. The linear measure of the length of an arc is found by the proportion:

$$\frac{\text{length of arc}}{\text{circumference}}=\frac{\text{central angle}}{360^\circ},$$

$$\text{or, arc}=\frac{\text{central angle}}{360^\circ}\times 2\pi R.$$

EXAMPLE: Find the length of an arc of 80° in a circle with a diameter of 12″.

SOLUTION:

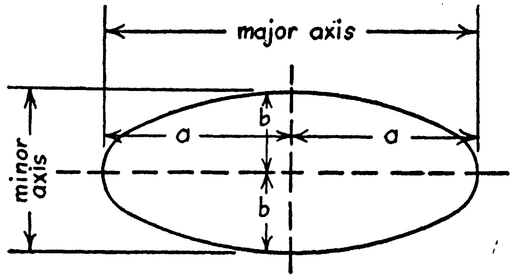
$$\text{arc}=\frac{80^\circ}{360^\circ}\times 2\times\frac{22}{7}\times 6,$$

$$\text{or, arc}=\frac{176}{21}=8.38'', \textit{ Ans.}$$

The Ellipse. An ellipse is a closed curve with a *major* and a *minor* diameter. It is perfectly symmetrical with diameter, or axis, and is not to be confused with an oval, which is egg-shaped, or narrower at one end than at the other. If the semi-major diameter= a, and the semi-minor diameter= b, then the length of the perimeter equals

$$P=\pi(a+b)K,$$

K being a constant depending upon the value of m, where $m=\frac{a-b}{a+b}$. The table shows the value of K for certain values of m:



m	K
0.1	1.002
0.2	1.010
0.3	1.023
0.4	1.040
0.5	1.064
0.6	1.092
0.7	1.127
0.8	1.168
0.9	1.216

EXAMPLE: Find the perimeter of an ellipse in which $a=5''$ and $b=3''$.

SOLUTION:

$$m=\frac{a-b}{a+b}=\frac{5-3}{5+3}=\frac{2}{8}=.25;$$

hence K=midway in value between 1.010 and 1.023, or 1.017 (by interpolation).

Therefore, perimeter $=\pi(a+b)K$

$$=\frac{22}{7}(5+3)\ (1.017)$$

$$=25.6'',\ \textit{Ans.}$$

If $a=b$, then $m=0$ and $K=1$, and the formula becomes $P=\pi(2a)$, or $P=\pi D$, which is what it must be, since if $a=b$, the ellipse has become a circle.

An alternative method for finding the perimeter of an ellipse involves the use of the formula $P=\pi\sqrt{2(a^2+b^2)}$, which also gives an approximate value only, although close enough for most practical purposes. Using this formula for the same problem above, we obtain:

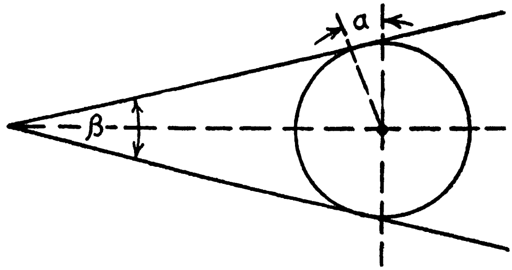
$$P=\pi\sqrt{2(25+9)}=\pi\sqrt{68}$$

$$=\frac{22}{7}\times 8.25=25.9'',\ \textit{Ans.}$$

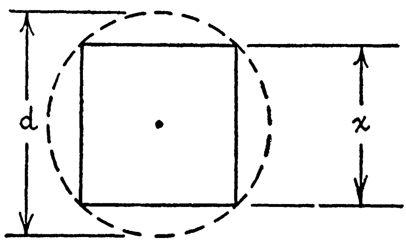
Exercise 60.

1. Find the length of a tangent drawn to a circle with a diameter of 8″ from a point 16″ from the center.

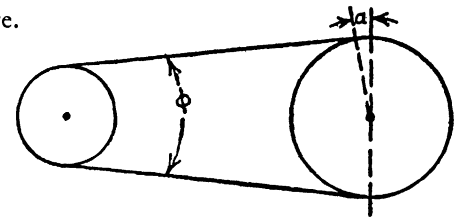
2. Find $\angle a$ if $\angle \beta =$ 34°20′; write a formula for $\angle a$ in terms of β. Assume that the radius to the point of tangency is perpendicular to the tangent.



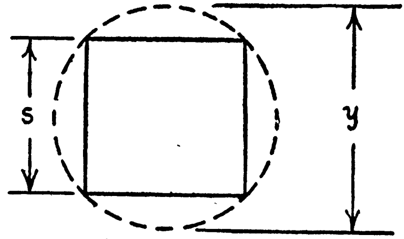
3. Find the side of a square inscribed in a 14-inch circle.
4. What is the value of x, if $d = 6.48''$? Write a formula for x in terms of d.



5. An electric motor is revolving at 1400 revolutions per minute. Through how many ft. does a point on the rim move in one second if the rim has a diameter of 36″?
6. An automobile wheel with tire has an overall diameter of 3½ feet. When traveling 20 miles per hour, how many revolutions is the wheel making per minute?
7. A square measures 6.4″ on a side. Find the diameter of a circle that will circumscribe this square.
8. If $\angle a = 28°45'$, find $\angle \phi$; write a formula for ϕ in terms of a.

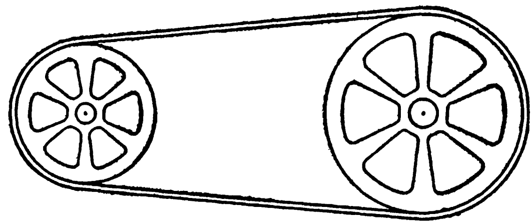


9. A square 10″ on a side is inscribed in a circle; find (a) the diameter, and (b) the circumference of the circle.
10. How many times must an automobile wheel with a 35-inch tire revolve in going half a mile?
11. Write a formula for finding y when S is known. What is the value of y when $S = .866''$?
12. The rear wheel of a carriage is twice as large in diameter as the front wheel. How much longer is its circumference?

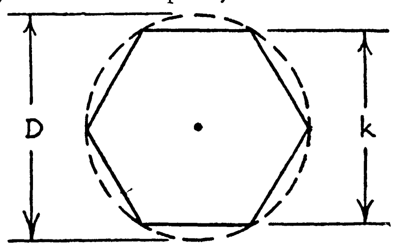


13. A circle 8″ in diameter is circumscribed about a square; find the distance across the flats of the square.

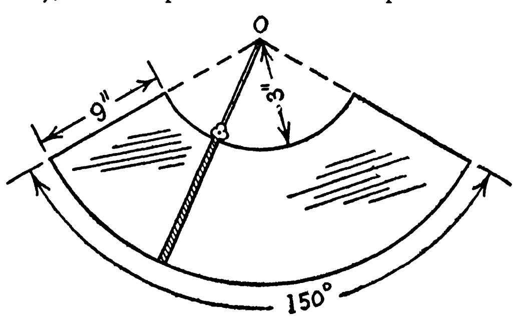
14. An arc of a circle measures 6.6″; if the diameter of the circle is 14″, find the central angle subtended by this arc.
15. A regular hexagon is inscribed in a circle with a diameter of 12″; find the distance across the flats of the hexagon.
16. Two pulleys in a machine shop are connected by a belt. One has a diameter of 18″ and the other a diameter of 12″. For each revolution of the large pulley, how many will the small pulley make?



17. Write a formula for finding k when D is known. What is the value of k when $D=3.12''$?



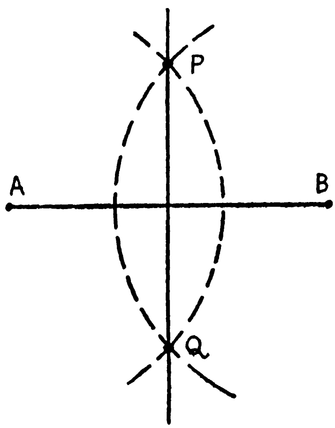
18. Through how many feet does the hub of a wheel travel during one revolution if the radius of the wheel is 8 ft.?
19. A squarehead nut is milled from round stock 1½″ in diameter; find the distance across the flats of the nut.
20. A flywheel 6 ft. in diameter turns at the rate of 120 revolutions per minute. What is the speed per minute of a point on the rim?
21. What diameter of round stock must be used to mill an hexagonal nut 1½″ across the flats?
22. A circular disc has a radius of 4.82″. How many inches in length is an arc on the edge of this disc if its subtended angle contains 156°?
23. A metal die is in the shape of an ellipse. If the major and minor axes are 6″ and 5″, respectively, find the perimeter of the ellipse.
24. How much further will one end of the windshield wiper travel than the other in swinging from left to right?

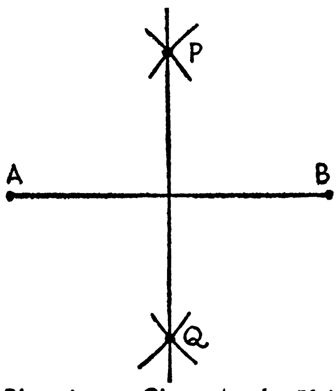


25. A thin cable is wrapped 14 times around the drum of a winch. If the diameter of the drum is 2½ ft., how many feet of cable are wound around the drum?

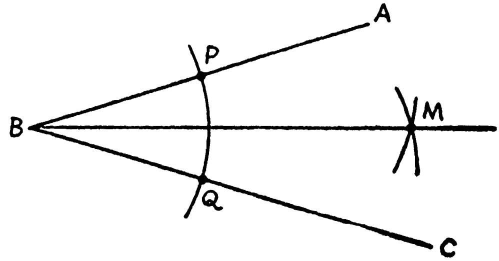
16. GEOMETRIC CONSTRUCTIONS

Bisecting a Given Line. By the term *geometric construction* is meant a drawing or layout made by using only a pair of compasses and a straight edge. A line may be bisected by swinging equal arcs from each end, and then connecting the points of intersection of the arcs by a straight line. Thus the ends of the line AB are used as centers to draw two arcs which intersect at P and Q; the line then drawn through points P and Q will bisect the line AB; it will also be perpendicular to it. Note that the radius used to swing the arcs must be greater than half the length of AB (why?); also, that it is not necessary to draw the arcs in full, but just enough to locate the points of intersection.

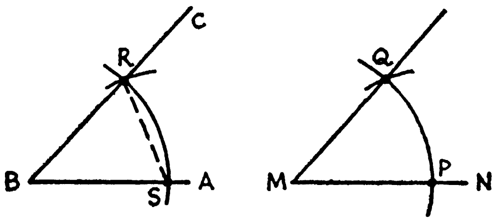




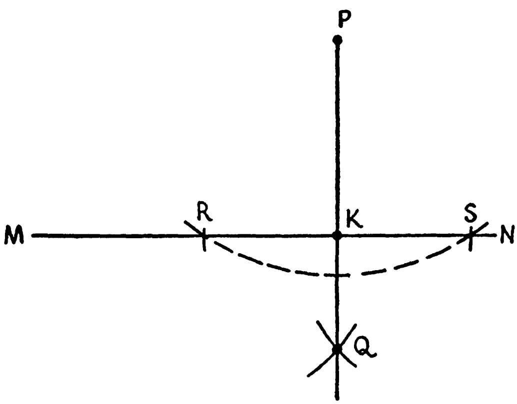
Bisecting a Given Angle. If it is desired to divide a given angle ABC into two equal parts, the vertex of the angle, B, is used as a center, and an arc is swung intersecting the sides of the angle in P and Q. Then using the points P and Q as centers, with any convenient radius, swing two equal arcs which intersect at M; draw a straight line connecting B with M. This line bisects $\angle ABC$.



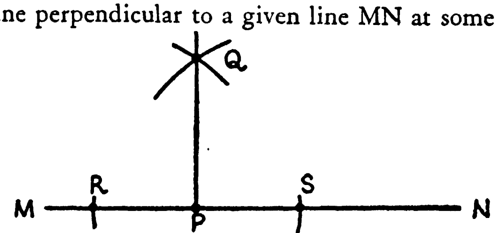
Constructing an Angle Equal to a Given Angle. If it is desired to transfer a given angle, i.e., construct an angle equal to a given angle, the procedure is as follows. Let ABC be the given angle which is to be transferred to the line MN. On ∠ABC, with B as a center, strike any convenient arc, intersecting the sides in R and S. With the same radius, and a center at M, strike another arc, intersecting MN in P. Then measure the *chord* RS; with this distance RS as a radius, and with P as center, strike another arc intersecting the larger arc in Q. Join M with Q. The angle NMQ equals the given angle ABC.



Erecting a Perpendicular from a Given Point outside a Given Line. If from the given point P it is required to construct a line perpendicular to the given line MN, take P as a center and a radius long enough to intersect MN in R and S. Then, with R and S as centers, respectively, strike two arcs with any convenient radius, intersecting in Q. Join P with Q. The line PQ is the perpendicular bisector of the segment RS (why?); hence PK is perpendicular to the given line MN, and, of course, passes through P, as required. (Note that PK does not necessarily bisect MN; also, that MN does not necessarily bisect PQ.)

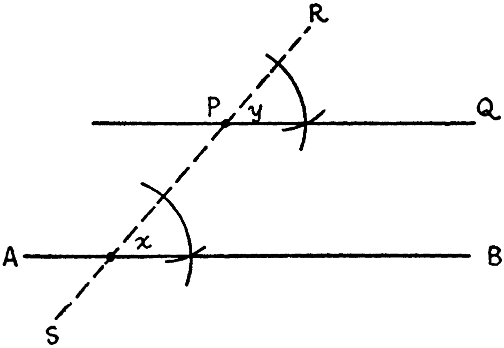


Erecting a Perpendicular at a Given Point on a Given Line. Should it be required to construct a line perpendicular to a given line MN at some particular point P *on* the line MN, the procedure is virtually the same. With P as center, and any convenient radius, strike two arcs intersecting MN in R and S. Then, with R and S as centers, and any convenient radius, strike arcs intersecting in Q;

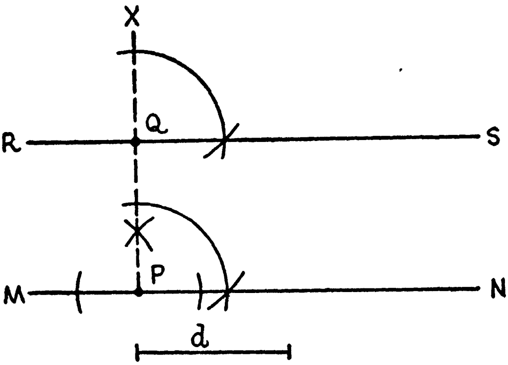


join P with Q. The line PQ is the required perpendicular. If the given point should happen to be at, or very near, either end of the given line, it would first be necessary to extend the line somewhat; then proceed as before.

Constructing a Line Parallel to a Given Line through a Given Point. It is desired to draw a line through P, and parallel to the line AB. Draw RS through P at any convenient angle to AB. Then construct $\angle y = \angle x$, using PR as one side of $\angle y$. The line PQ will be parallel to AB, and, of course, passes through P, as required.

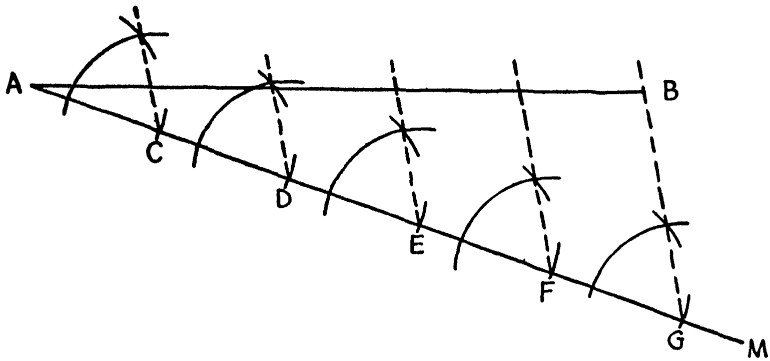


Constructing a Line Parallel to a Given Line at a Given Distance from the Line. It is desired to construct a line which will be parallel to the given line MN, and also at a given distance, *d*, from MN. The procedure follows. At any convenient point P on MN, erect a perpendicular PX, as described above. Then on PX, beginning at P, measure off a length PQ equal to the given distance *d*. Now through Q construct the line RS parallel to MN (by constructing a right angle at Q, using QX as one side). The line RS is the required line, for it is not only parallel to MN, but is also at the given distance *d* from MN.



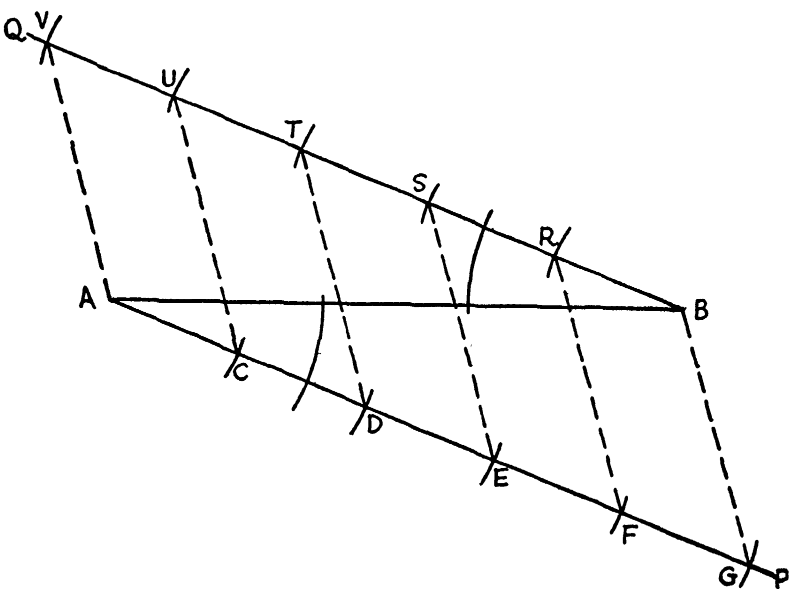
Dividing a Given Line into Any Number of Equal Parts. If it is desired to divide a line of given length into any number of equal parts, either of the following methods may be used.

(a) Suppose the line AB is to be divided into five equal segments; draw line AM at any angle to AB, and long enough to lay out the required number of divisions conveniently. Step off five equal segments on line AM,



starting at A; connect the last point (G) with point B. Then draw lines through C, D, E and F parallel to BG. The intersections of these lines with AB will divide it into the required number of equal parts.

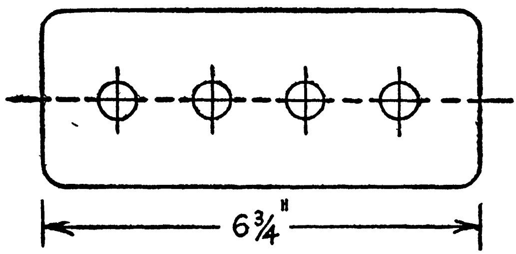
(b) The alternative method consists of drawing two lines, AP and BQ. parallel to each other, through A and B respectively, at any convenient

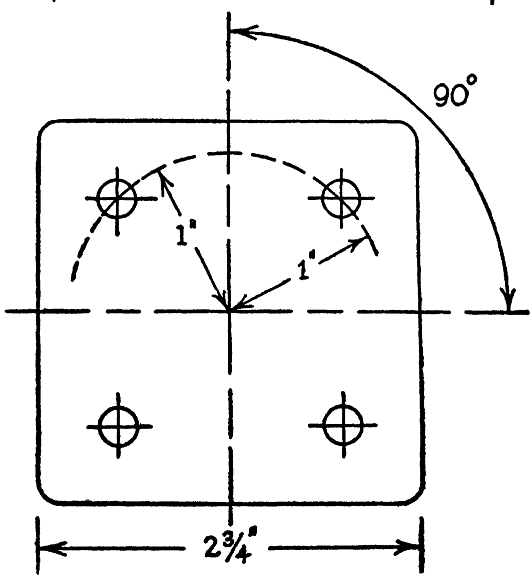


angle to AB. Then lay off the same number of equal segments on AP and BQ, starting in each case from A and B, respectively. Connect AV, CU, DT, ES, FR, and GB. The intersections of these lines with the line AB will divide it into the required number of equal parts.

Exercise 61.

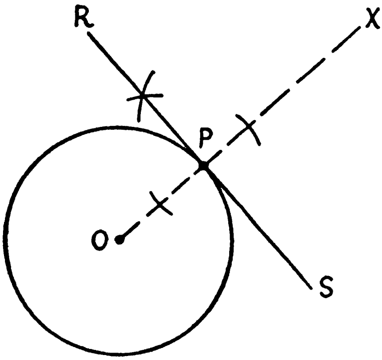
1. Draw a horizontal line 4⅝″ long, and bisect it geometrically; do the same for a vertical line 5³⁄₁₆″ long.
2. Lay out an angle of 70° with a protractor; then bisect the angle, showing all construction lines.
3. With a protractor, draw an angle of 110°; bisect this angle geometrically.
4. Lay out an angle of 27° with a protractor and bisect this angle.
5. Erect a perpendicular to a line 3¾″ long at a point 1½″ from either end of the line.
6. Erect a perpendicular to a line 3⅞″ long at either end of the line. (*Hint:* Extend the line first.)
7. Draw an oblique line 4¼″ long; then mark a point, anywhere, approximately 2″ from either side of the line. Now construct a perpendicular to the line from that point.
8. Draw a line 5⅝″ long and divide it into 7 equal parts.
9. Four equally spaced holes are to be drilled in the plate shown; the end holes are to be as far from the edge as the distance between the holes. Locate the centers by means of a geometric construction.
10. Construct an angle of 90°. Then construct an angle of 45° by bisecting the right angle. Construct an angle of 22½° by bisecting the 45° angle.
11. Lay out (with the protractor) an angle of 60°. Now construct, by bisection, an angle of 30°; of 15°; of 7½°.





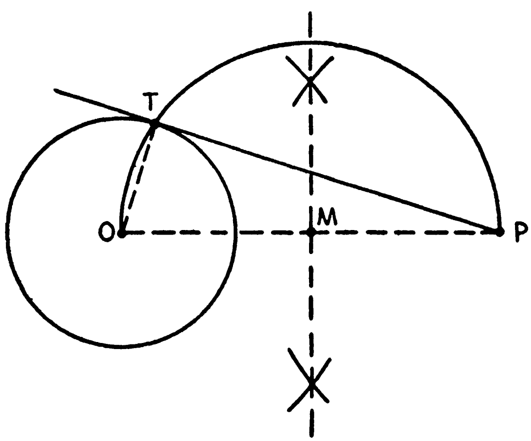
12. Draw an oblique line. Construct a line parallel to this line and at a distance of 1⅞″ from it.
13. Construct a square, 2¾″ on a side; locate its center. Then locate the centers of the four holes as indicated.
14. Draw any acute triangle. Construct (a) an altitude to the longest side; (b) a medium to the shortest side.
15. Construct a right triangle whose short sides are 1½″ and 2½″. Construct (a) the perpendicular bisectors of all three sides; (b) the altitude upon the hypotenuse.

Constructing a Tangent to a Circle at a Given Point on the Circle.

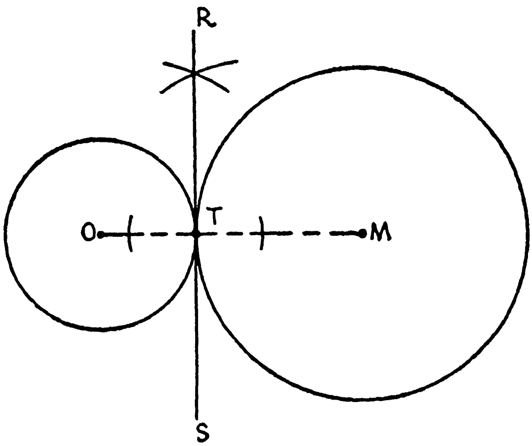


To the given circle O, it is required to construct a tangent touching the circle at the given point P. *Construction:* Draw the radius OP, and extend OP a convenient distance beyond the circle to X. Construct the line RS perpendicular to OX at P. The required tangent to circle O at point P is the line RS.

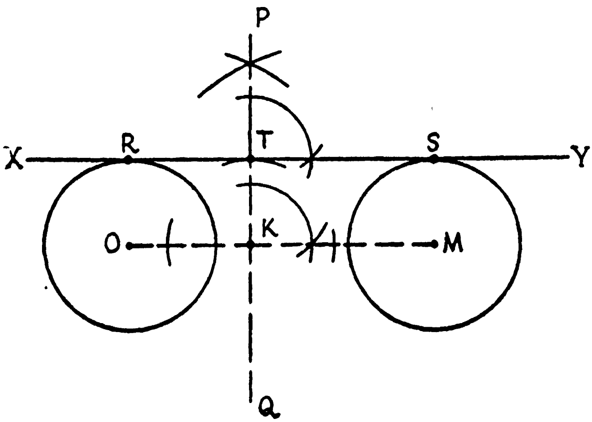
Constructing a Tangent to a Given Circle from a Given Point outside the Circle. *Construction:* Join the center O of the given circle with the given point P. Determine the midpoint M of OP by constructing the perpendicular bisector of OP. With M as center and MO as a radius, construct a semicircle on OP as diameter, intersecting the given circle in point T. Join T with P, and extend PT, which is the required tangent to the circle from the given point P. (why?) (*Hint:* What can you say of ∠OTP?) The other tangent to the circle from P could of course be constructed in the same way by completing the circle on OP as diameter.



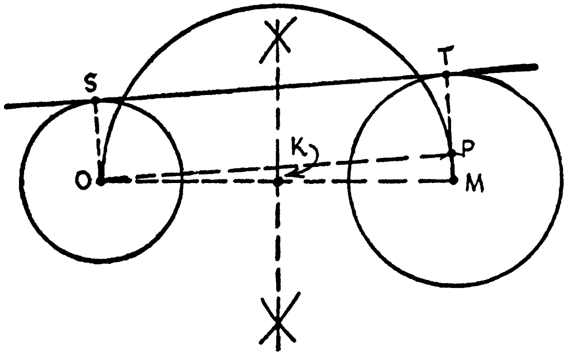
Constructing a Common Tangent to Two Externally Tangent Circles. *Construction:* Join the centers O and M of the two given tangent circles. This line of centers will pass through the point of tangency, T. At T, construct a perpendicular to OM. The line RS is the required common tangent to the two circles.



Constructing a Common External Tangent to Two Given Equal Circles. *Construction:* Draw the line of centers, OM. At any convenient point on OM, say K, erect a perpendicular to OM, say PQ. On PQ, beginning at K, lay off a distance KT equal to the radius of either of the two equal circles. Through T construct a line XY parallel to OM (or perpendicular to PQ). The line XY will be tangent to the circles at R and S respectively, as required.

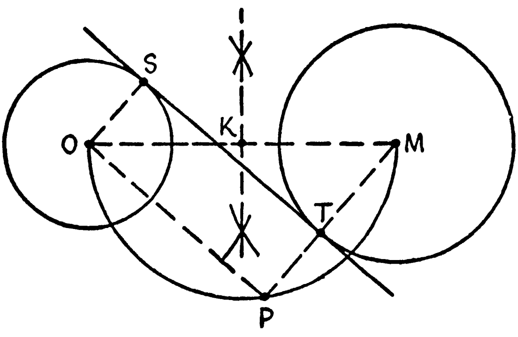


Constructing a Common External Tangent to Two Given Unequal Circles. *Construction:* Draw the line of centers OM; bisect OM in K.



With K as center and OK as radius, construct a semicircle on OM as diameter. Then with M as center, and a radius equal to the *difference between the lengths of the radii of the given circles* $(R—r)$, strike an arc intersecting the semicircle in P. Join M with P and extend to T, the intersection with the circle. Draw OP. Now construct a line through T parallel to OP and extend it; it will touch the other circle at S, and ST is the required external tangent. An alternative procedure would be, instead of constructing TS parallel to OP, to erect a perpendicular to OP at O, and extend this perpendicular to the circle at S; join S with T.

Constructing a Common Internal Tangent to Two Given Unequal Circles. *Construction:* Draw the line of centers OM; bisect OM in K. With K as center and OK as radius, construct a semicircle on OM as diameter. Then with M as center, and a radius equal to *the sum of the lengths of the radii of the given circles* $(R+r)$, strike an arc intersecting the semicircle in P. Join M with P, and let MP intersect the given circle in T. Draw OP. Now construct a line through T parallel to OP and extend it; it will touch the other circle in S, and ST is the required internal tangent.



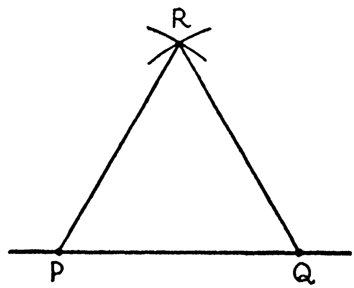
An alternative procedure would be, instead of constructing TS parallel to OP, to erect a perpendicular to OP at O, and extend this perpendicular to the circle at S; join S with T.

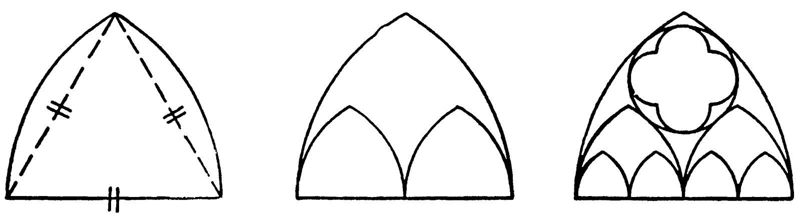
Exercise 62.

1. Draw a circle having a radius of 2½″; select any point on the circle and construct a tangent to the circle at that point.

2. Draw a circle with a radius of 1⅞″ and extend any radius to a point P such that P is 3¼″ from the center. Construct a tangent to the circle from P.
3. Draw two externally tangent circles, one with a radius of 1½″, the other with a radius of 2¼″.
4. Construct two circles, each with a radius of ¾″, having their centers 3″ apart. Construct a common external tangent to the two circles.
5. Show by construction how two pulleys each with a 3″ diameter are connected (a) by an open belt; (b) by a crossed belt.
6. Show by construction how a 2″ pulley and a 3½″ pulley are connected by (a) an open belt; (b) a crossed belt.
7. Draw any obtuse triangle. Construct (a) the perpendicular bisectors of the three sides; (b) the circumscribed circle of the triangle.

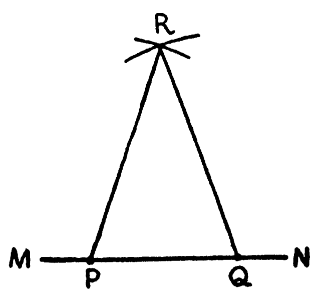
Constructing an Equilateral Triangle. If it is desired to construct an equilateral triangle with a given length as its side, all that needs be done is to lay off a segment PQ equal to the required side; then, with P and Q as centers, respectively, and with PQ as a radius, strike off two arcs intersecting in R. Join R with P and with Q; the triangle PQR is the required equilateral triangle. This construction is the basis of the well-known Gothic arch of medieval times, commonly used in cathedrals and other public buildings.



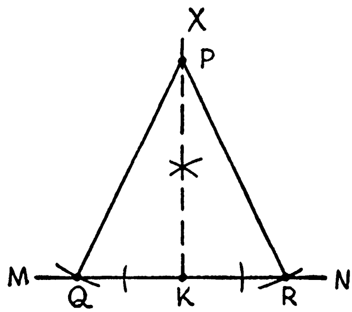


Constructing an Isosceles Triangle. Several possibilities may arise.

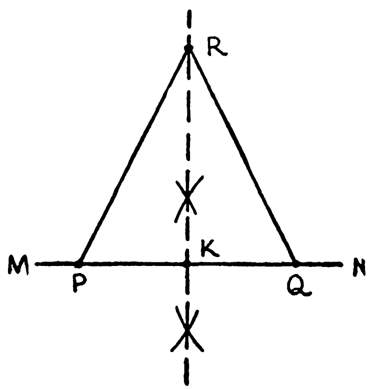
(1) *Given the base and the length of each of the two equal sides.* On any line MN lay off the required base, PQ. With P and Q as centers, respectively, and a radius equal to the given equal sides, strike two arcs intersecting in R. Triangle PQR is the required isosceles triangle.



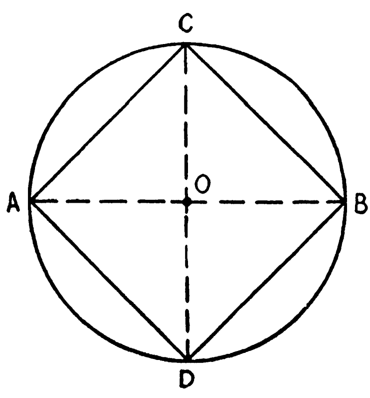
(2) *Given the altitude and the two equal sides.* At any convenient point K on any line MN, erect a perpendicular to MN and extend it to X. On KX, beginning at K, lay off a distance KP equal to the required altitude. With P as a center and a radius equal to the length of the required equal sides, strike two arcs intersecting MN in Q and R. Triangle PQR is the required isosceles triangle.



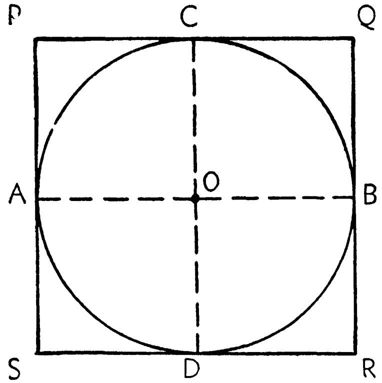
(3) *Given the base and the altitude.* On any line MN lay off the required base, PQ. Bisect PQ in K; at K erect a perpendicular to MN. Beginning at K, lay off a distance KR equal to the required altitude. Join R with P and with Q. Triangle PQR is the required isosceles triangle.



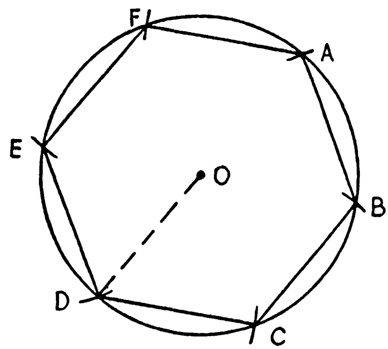
Inscribing a Square in a Given Circle. Draw any convenient diameter AB; then construct another diameter, CD, perpendicular to AB. Join points A, C, B, and D; the resulting figuré is the required inscribed square.



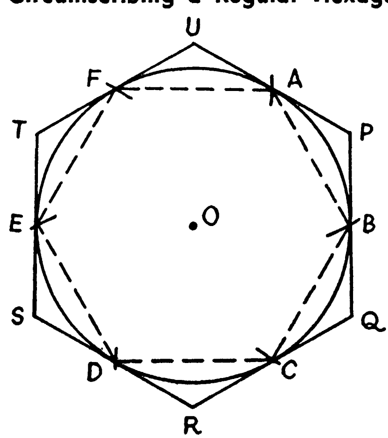
Circumscribing a Square about a Given Circle. Construct two perpendicular diameters, AB and CD. At each of the four extremities A, B, C, and D construct a tangent to the circle. The figure PQRS is the required circumscribed square.



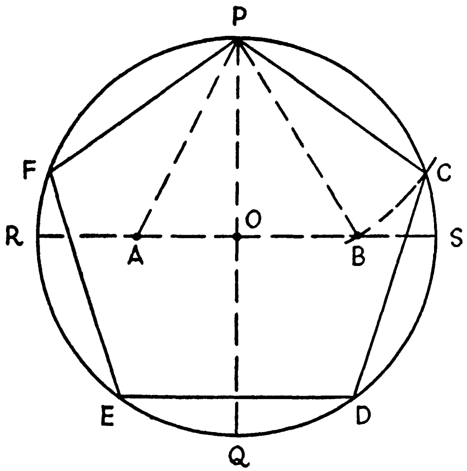
Inscribing a Regular Hexagon in a Given Circle. Starting with any point A on the given circle, and using a radius equal to the radius OA of the given circle, strike off successive arcs at B, C, D, E and F. Join the six points A, B, C, D, E and F; the resulting figure is the required inscribed hexagon. Each side of this hexagon is equal in length to the radius of the circle in which it is inscribed.



Circumscribing a Regular Hexagon about a Given Circle. Divide the given circle, as above, into six equal arcs, the points of division being A, B, C, D, E and F. At each of these points of division construct a tangent to the circle. The figure PQRSTU is the required circumscribed hexagon.



Inscribing a Regular Pentagon in a Given Circle. Construct any two perpendicular diameters, such as RS and PQ. Now bisect RO, and join the midpoint A with point P. With a radius equal to PA, lay off AB=PA; join P and B. Then with a radius equal to PB, and with P as center, strike an arc BC, cutting the circle in C. The chord PC is the side of the required pentagon; step off arc PC five times, beginning at P, and the figure PCDEF will be the desired regular inscribed pentagon.



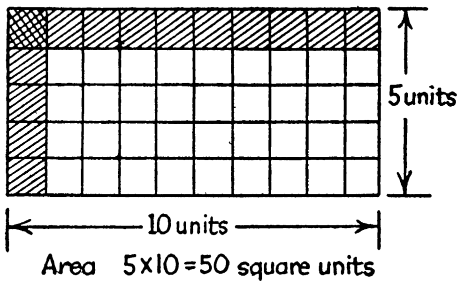
Exercise 63.

1. Construct an equilateral triangle 1½″ on a side.
2. Construct an angle of 60°; by bisection, construct an angle of 30°; 15°.
3. Show how to construct (with a protractor) an angle of 120°; 75°; 105°; 67½°; 135°; 150°; 165°.
4. Construct an isosceles triangle having a base 2½″ and its equal sides each 3½″ long.
5. Construct an isosceles triangle with a base of 3″ and an altitude of 1¼″.
6. Construct an isosceles triangle with an altitude of ¾″ and equal sides of 1½″ each.
7. Construct a Gothic arch on a base line of 1½″.
8. Construct an equilateral triangle 1⅜″ on a side; then construct the inscribed and circumscribed circles.
9. Construct an isosceles triangle with a base 2¼″ long and base angles of 30° each.
10. Construct an isosceles triangle with equal sides of 1″ each and a vertex angle of 120°.
11. Construct a 30°—60°—90° triangle with an hypotenuse of 2⅝″.
12. Construct an isosceles right triangle 1¼″ on each of the two equal sides.
13. Construct the largest possible square in a 1¾″ circle.
14. Show by construction the largest square which can be milled from a piece of 1½″ round stock.
15. Show by construction the largest hexagon that can be milled on a 2″ round shaft.

16. Draw a circle with a radius of 1½″. Inscribe a rectangle in this circle, having a width of 1″.
17. Construct a 30°—60°—90° triangle such that its shortest side is 1½″ long; now construct its inscribed and circumscribed circles.

17. MEASUREMENT OF AREAS

Area of Squares and Rectangles. Surface measure, or the measure of area, is expressed in square units, such as the square inch, the square foot, or the square centimeter. Thus the area of a plane geometric figure may be described as the number of units of surface measure contained in the figure in question. A square inch is defined as a square, each of whose sides measures one linear inch; similarly for a square foot, etc. It is readily seen that the area of a rectangle may be found simply by multiplying the length of the rectangle by its width; or, expressed as a formula:

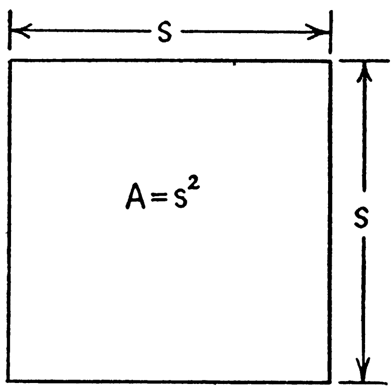


$$A=lw.$$

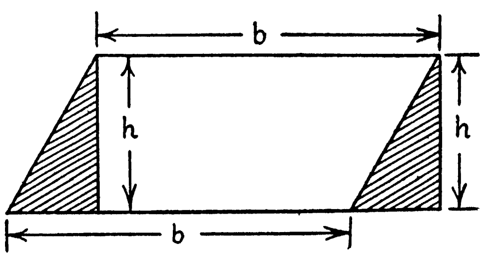
For a square, this becomes

$$A=s\times s,$$

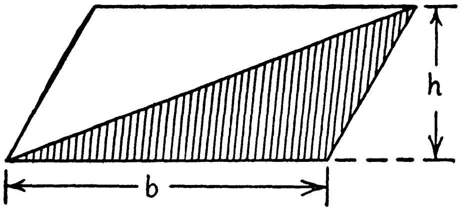
or $A=s^2$.



Area of a Parallelogram. In the case of a parallelogram, the area is found by multiplying the base by the altitude, since from the diagram it is clear that the shaded triangles are equivalent in area; also, the parallelogram is equivalent in area to the rectangle, which has the same base and altitude, respectively, as the parallelogram. Hence the area of a parallelogram is $A=bh$.



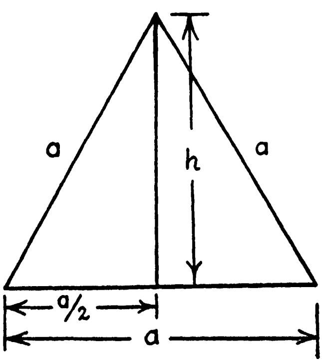
Area of a Triangle. When it is remembered that a parallelogram is divided into two equivalent triangles by either of its diagonals, one method of computing the area of a triangle is readily suggested. The base of the shaded triangle is equal to the base of the parallelogram, and the altitude of this triangle is equal to that of the parallelogram; therefore, since the area of the triangle is half that of the parallelogram, the area of the triangle is half of the base times the altitude; or, $A=\frac{1}{2}bh$.



To use this formula it is necessary to know the length of the altitude drawn to one of the three sides, as well as the length of that side. This may not always be known or conveniently measured. If, for example, we know only the lengths of each of the three sides, the following formula may be used instead:

$A=\sqrt{(s)\ (s-a)\ (s-b)\ (s-c)}$, where a, b, and c represent the lengths of the three sides, respectively, and s equals the *semiperimeter*, or $\frac{1}{2}(a+b+c)$. Thus, if the sides of a triangular sheet of metal measure 8″, 12″, and 16″, the area of the sheet equals $A=\sqrt{(18)\ (18-8)\ (18-12)\ (18-16)}=\sqrt{(18)\ (10)\ (6)\ (2)}=\sqrt{2160}=$ 46.4 sq. in. As a special case, the area of an equilateral triangle yields another formula; since in this case $a=b=c$, the above formula becomes:

$$A=\sqrt{\left(\frac{3a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)}=\frac{a^2}{4}\sqrt{3},$$

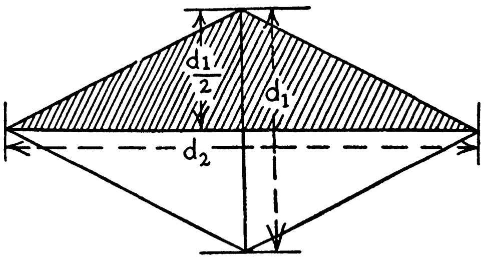


where a represents any one of the three equal sides. The same result can also be obtained by using the altitude and base directly, as follows:

$$h=\frac{a}{2}\sqrt{3};$$

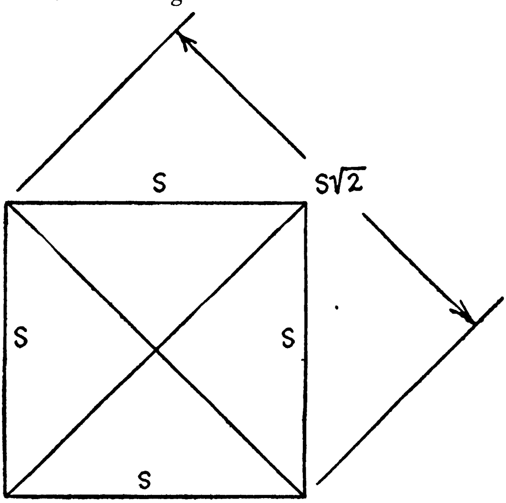
but $A=\frac{1}{2}(\text{base})\times(\text{altitude})$; therefore $A=\frac{1}{2}(a)\left(\frac{a}{2}\sqrt{3}\right)$, or $A=\frac{a^2}{4}\sqrt{3}$, as before. Thus, if the side of an equilateral triangle is 6″, its area equals $\frac{36}{4}\times\sqrt{3}=9\times1.73=15.6$ sq. in.

Area of a Rhombus in Terms of the Diagonals. Since the diagonals of a rhombus not only bisect each other, as in all parallelograms, but inter-

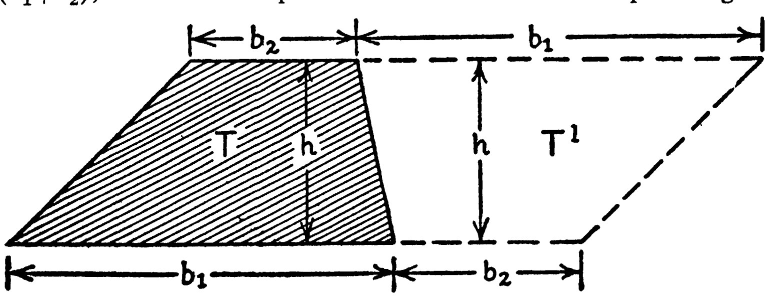


sect at right angles as well (which is true only in a rhombus or a square, i.e., an equilateral parallelogram), then the base of the shaded triangle $=d_2$ and its altitude$=\frac{d_1}{2}$; hence the area of the shaded triangle is ½ (d_2) $\left(\frac{d_1}{2}\right)$, or ¼ d_1d_2. Therefore the area of the entire rhombus (being twice as large as the shaded triangle) is given by: $A=2(\frac{1}{4}d_1d_2)$, or $A=\frac{1}{2}d_1d_2$.

An interesting illustration of this relation is found by applying it to a square, where each diagonal equals the other, and is given (as we have already seen) by $d=s\sqrt{2}$, where s represents the side of the square. Applying the above formula, then, we obtain: A (of a square)$=\frac{1}{2}(s\sqrt{2})\times(s\sqrt{2})$, or $A=s^2$, as before.



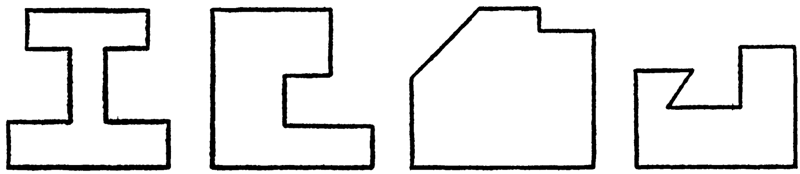
Area of a Trapezoid. The area of the entire parallelogram is evidently $h(b_1+b_2)$; since either trapezoid is half the area of this parallelogram,



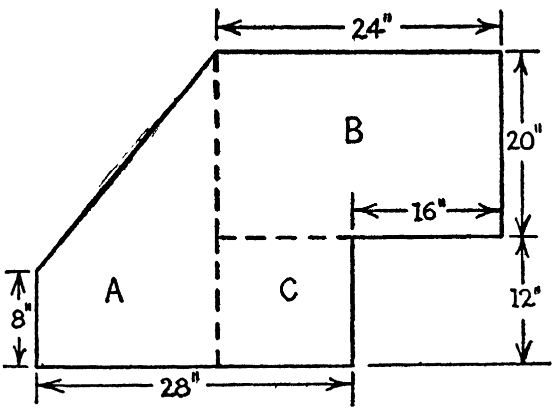
then the area of the trapezoid must be given by $A=\frac{1}{2}h(b_1+b_2)$. Since the median of a trapezoid equals one half the sum of its bases, the area of a trapezoid is also equal to the altitude times the median; or,

$$A=hm, \text{ where } m=\tfrac{1}{2}(b_1+b_2).$$

Irregular Structural Shapes. Metal plates and other structural parts are often in the shape of irregular polygons as here shown. If there are no



rounded corners or other curved lines in the figure (which is frequently the case, however) the area may be computed by decomposing the figure into convenient rectangles, triangles, trapezoids, etc., taking the appropriate measurements, finding the area of each part, and then simply adding them together.

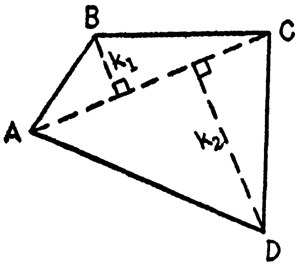


EXAMPLE: Find the area of the iron plate shown, with dimensions as given.

SOLUTION: Break up the original figure into trapezoid A and rectangles B and C.

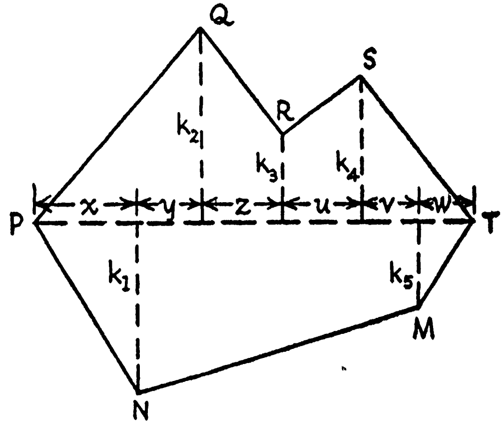
Trapezoid	$A=\frac{1}{2}h(b_1+b_2)=\frac{1}{2}(20)(8+32)$		$=400$
Rectangle	$B=$	24×20	$=480$
Rectangle	$C=$	12×8	$=96$
	Total figure		$=976$ sq. in., *Ans.*

A somewhat more general method, particularly for irregularly shaped figures, consists of drawing a convenient diagonal, dropping and measuring the perpendiculars to this diagonal from each remaining vertex, and measuring the projections of these perpendiculars upon the diagonal as well; with this data it is a simple matter to find the area of each of the separate triangles and trapezoids into which the figure has been decomposed, as suggested below.



Area of ABCD

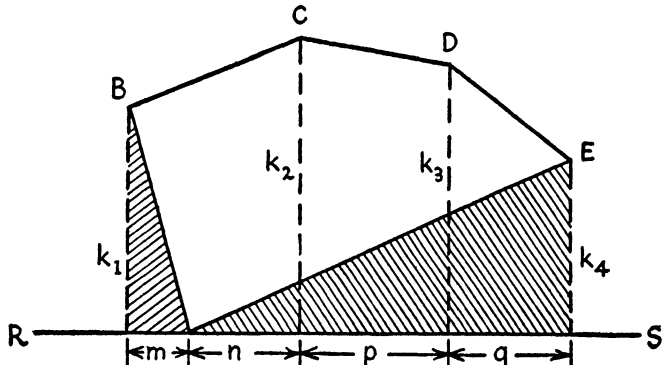
$=\frac{1}{2}(AC)k_1+\frac{1}{2}(AC)k_2$
$=\frac{1}{2}(AC)(k_1+k_2)$



Area of MNPQRST

$=\frac{1}{2}(x+y)k_2+\frac{1}{2}(z)(k_2+k_3)+$
$\frac{1}{2}(u)(k_3+k_4)+$etc.

Usually the modified method of using a base line (RS) instead of the diagonal is slightly more convenient. Thus the area of ABCDE is given by the sum of the areas of the three trapezoids diminished by the sum of the areas of the two triangles; or,

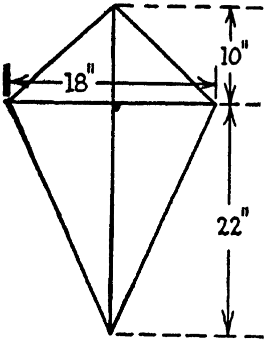


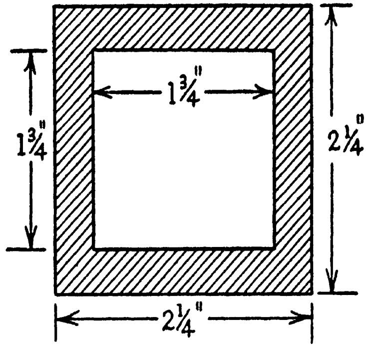
$$A=\frac{1}{2}(m+n)(k_1+k_2)+\frac{1}{2}(p)(k_2+k_3)$$
$$+\frac{1}{2}(q)(k_3+k_4)-\frac{1}{2}mk_1$$
$$-\frac{1}{2}(n+p+q)(k_4).$$

Exercise 64.

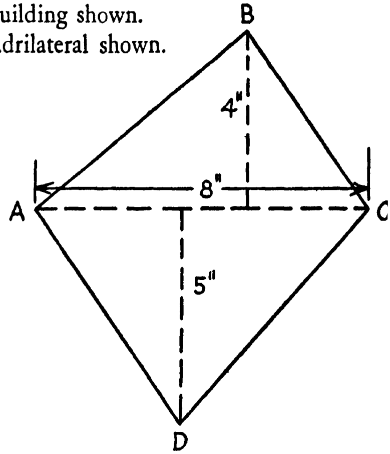
1. Find the area of a 30°—60°—90° triangle whose hypotenuse is 14″.
2. Find the area of an isosceles triangle whose base is 6″ and whose equal sides are each 8″ long.

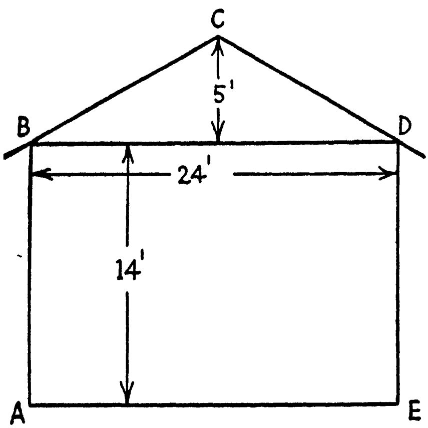
3. The sides of a parallelogram are 6″ and 12″, and its diagonal is 16″. What is its area?

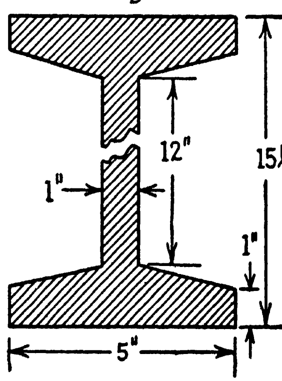




4. How many square inches of paper are needed to cover this kite frame?
5. What is the area of a fiber gasket cut in the dimensions shown?
6. Find the area of the front of the building shown.
7. Find the area of the irregular quadrilateral shown.

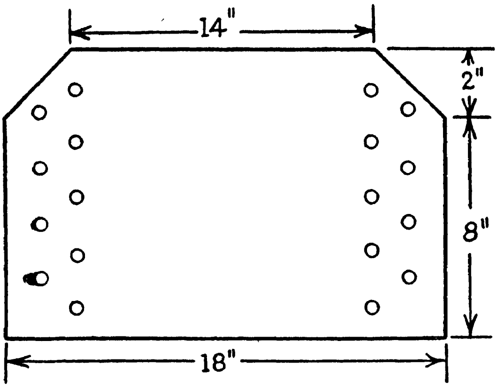


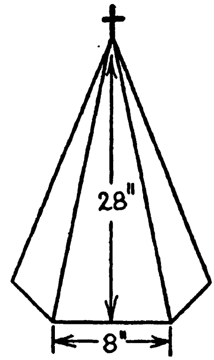




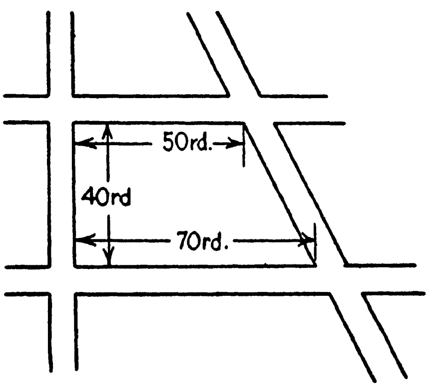
8. What is the cross-sectional area of the beam shown in the sketch?

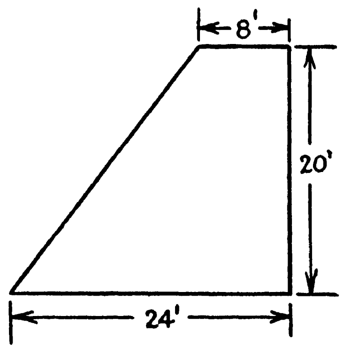
9. Find the total lateral surface of this 8-sided church steeple.
10. A steel plate riveted to a girder has the shape and dimensions shown. Find its area.



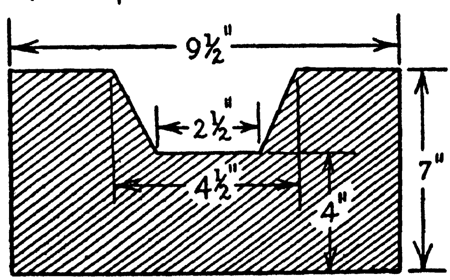


11. A plot of land is bounded by intersecting roads as shown. (a) How many square rods does the plot contain? (b) How many acres is this? (One acre=160 sq. rd.)
12. The cross section of the dam of a reservoir has the dimensions shown. What is the surface area of the end of the dam?

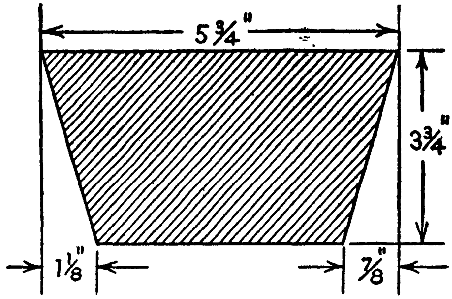


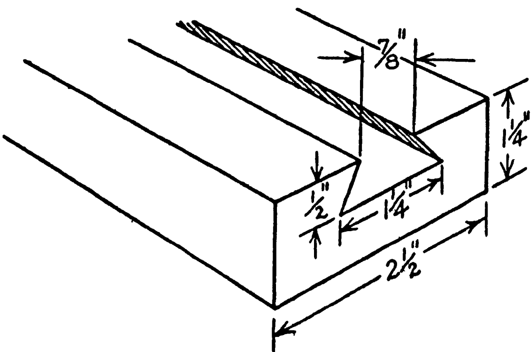


13. Find the area of the cross section of the steel channel beam shown.



14. The cross section of a fixture on a lathe is a trapezoid with the dimensions shown; what is the area of its face?



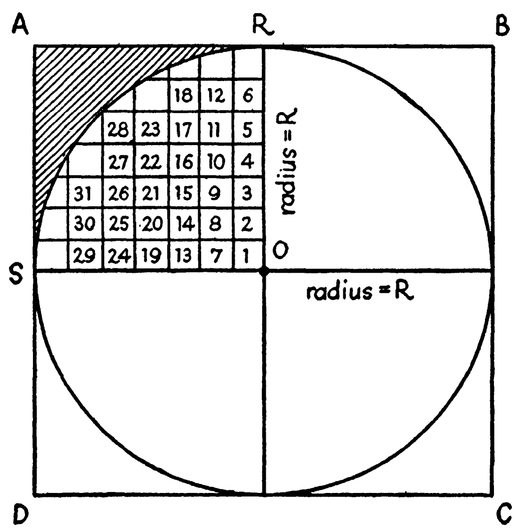


15. Find the cross-sectional area of the wooden dovetail shown in the figure.

Area of a Circle. The area included by a circle is found by squaring the radius and multiplying by 3 1/7; i.e.,

$$A=\pi R^2, \text{ or, since } R=\frac{D}{2},\ A=\frac{\pi D^2}{4}=.7854D^2.$$

Each side of the square is 14 units long; so is the diameter. The radius is 7 units in length. Now count the number of small squares in the quadrant ORS. There are 31 whole squares. Try to estimate the equivalent number of whole squares in the remaining imperfect squares. If you are careful, you will probably estimate 7 or 8 more; if you are very skillful, you will obtain approximately 7½ more, or about 38½ in all. Now, since a



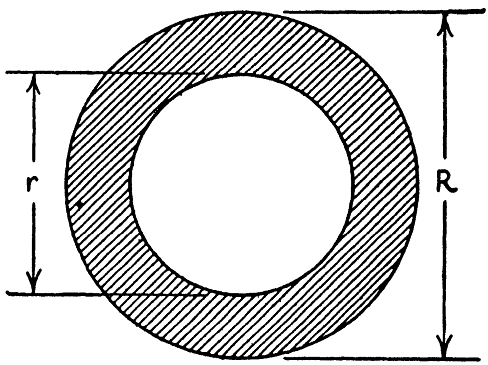
quadrant is ¼ of a circle, the entire circle contains 4×38½, or about 154 square units. Let's call the area of the circle A. Since R=7, then $R^2=49$. By counting the small squares, we found that A=154. Now divide 154 by 49:

$$\frac{A}{R^2}=154/49=3\tfrac{1}{7}=\pi, \text{ showing}$$

$$\text{that } \frac{A}{R^2}=\pi, \text{ or } A=\pi R^2, \text{ as above.}$$

Area of a Ring. The area included between two concentric circles is called a *ring,* and is obviously found by deducting the area of the smaller circle from that of the larger, i.e.,

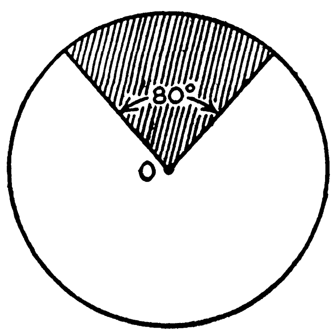
$$A=\pi R^2-\pi r^2, \text{ or}$$
$$A=\pi(R^2-r^2).$$



Area of a Sector. A sector of a circle is a figure bounded by a central angle and its subtended arc. The area of a sector (like the length of an arc) may easily be found by proportion; for

$$\frac{\text{Area of Sector}}{\text{Area of Circle}}=\frac{\text{Central Angle}}{360°};$$

$$\text{or, Sector}=\frac{\text{Central Angle}}{360°}\pi R^2.$$

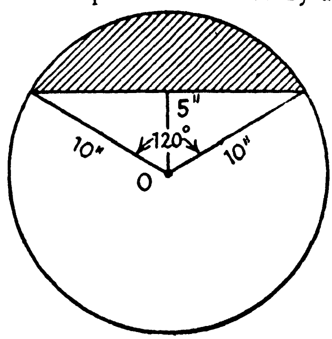


EXAMPLE: Find the area of an 80°-sector in a circle with a diameter of 7 inches.

SOLUTION: $R=3\tfrac{1}{2}$

Sector=(80/360) (22/7) (7/2) (7/2)= 8.56 sq. in., *Ans.*

Area of a Segment. A segment of a circle is that portion bounded by a chord and its subtended arc. Hence its area may be found by deducting the area of the triangular portion from the entire area of the corresponding sector.

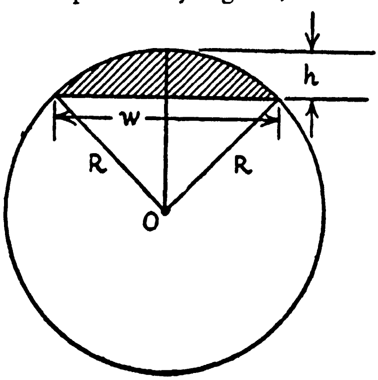


EXAMPLE: In a circle with a 10″ radius, find the area of a segment whose arc is 120°.

SOLUTION:

$$\text{Sector} = (^{120}/_{360})(3.14)(100) = 104.7$$
$$\text{Triangle} = {}^{100}/_{4}\sqrt{3} = 43.3$$
$$\text{Segment} = \text{Sector} - \text{Triangle} = 61.4 \text{ sq. in., } \textit{Ans.}$$

If the span w only is given, as well as the radius, but not the central angle, the procedure is as follows: First the value of the height h is determined from either of the following formulas, whichever is more convenient:



(1) $h = R - \sqrt{R^2 - (\frac{1}{2}w)^2}$

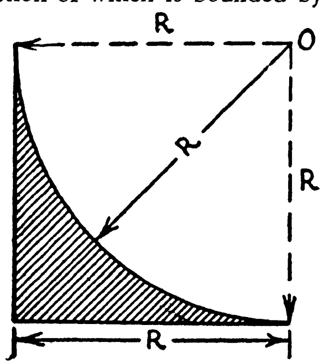
(2) $h = \frac{1}{2}(D - \sqrt{D^2 - w^2})$

The area may then be found by using the formula:

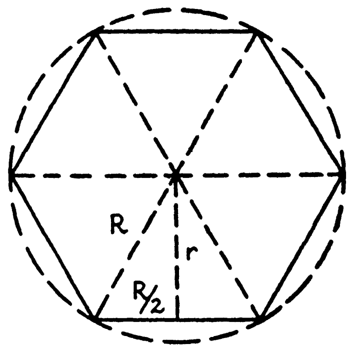
$$A = \frac{h(\frac{4}{3}w^2 + h^2)}{2w}$$

A *fillet* is a structural piece, the cross section of which is bounded by two adjacent sides of a square and the quadrant whose center is the opposite vertex of the square, as shown in the accompanying figure. The area of such a cross section is simply obtained by subtracting the area of the sector, which is one-fourth that of the circle, from the area of the square; thus

$$A = R^2 - \frac{1}{4}\pi R^2.$$

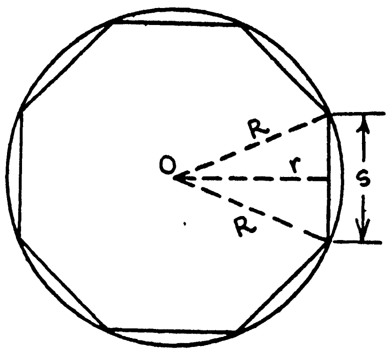


Area of Regular Polygons. The area of the square and the equilateral triangle have already been discussed. The only other regular polygon frequently encountered in the machine shop is the regular hexagon. Its area is simply six times that of one of the six congruent equilateral triangles of which it is made up, each side of the regular hexagon being equal to the radius of the circumscribed circle; thus



$$A=\frac{3R^2\sqrt{3}}{2}.$$

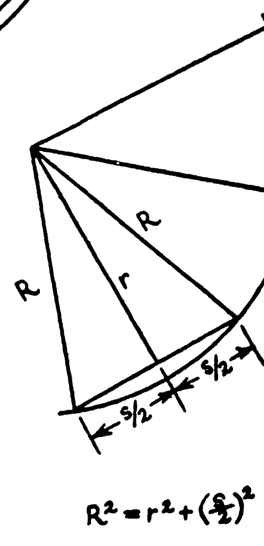
A general way of expressing the area of any regular polygon is in terms of its sides and apothem: $A=\frac{1}{2}(ns)r=Pr$,



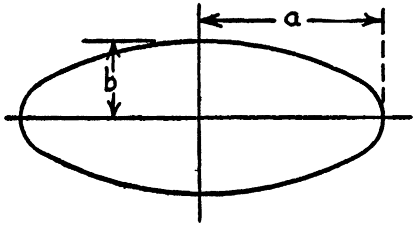
where n=number of sides, s=length of each side, and r=the apothem. Of course, if any two of the measurements R (radius), r (apothem) and s (length of side) are known, the third could always be found by the right-triangle rule. This is not always too convenient, however, and trigonometric methods are more practical, as will be seen later. Thus, since

$$r=\sqrt{R^2-\frac{s^2}{4}}, \text{ then}$$

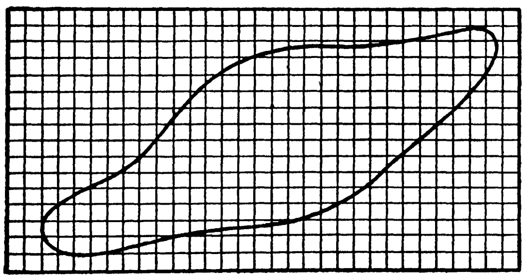
$$A=\frac{1}{2}(ns)\sqrt{R^2-\frac{s^2}{4}}.$$

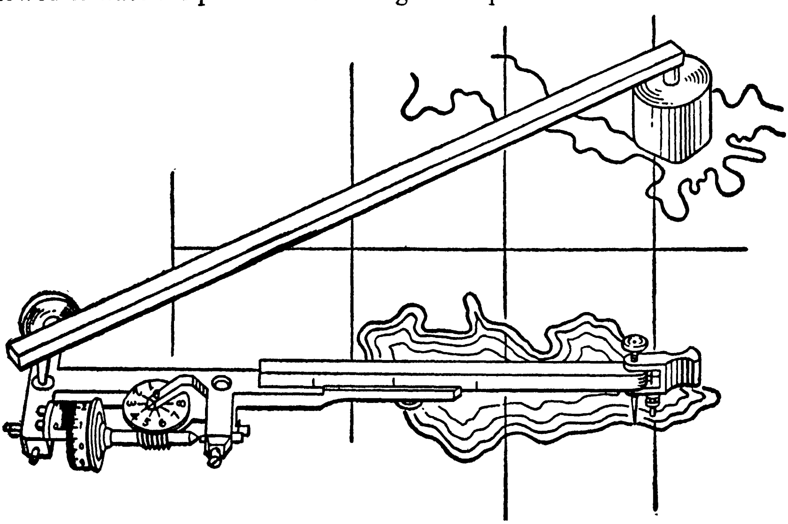


Area of an Ellipse. The area enclosed by an ellipse is found by multiplying the product of the major axis by the minor axis by π; or, $A=\pi ab$. If $a=b$, the ellipse becomes a circle, and $A=\pi aa=\pi a^2$, as already seen.

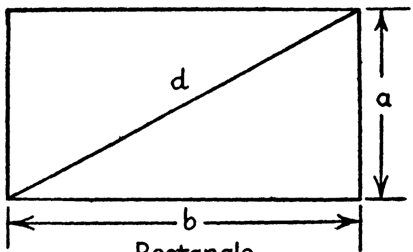


Irregular Areas. Not infrequently it is desired to obtain the area of an irregularly shaped figure, such as a plot of ground, the cross section of a specimen of material, a steam engine indicator, etc. Two methods are available in such cases: (1) an approximate value of the area may be found by drawing the figure to scale against a ruled area, and then counting and estimating the number of square units included within the area. If a more accurate value of the area is required, a *planimeter* may be used. This is a mechanical device which computes the area automatically as the needle of the instrument is allowed to trace the perimeter of the figure in question.



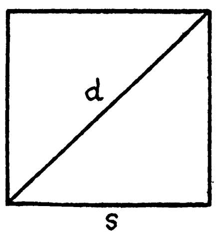


SUMMARY OF FORMULAS



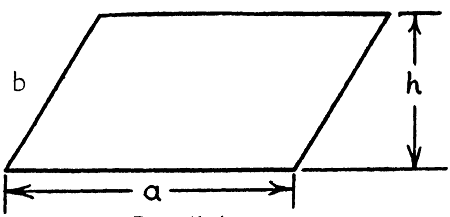
Rectangle

$P=2(a+b)$
$A=ab$
$d=\sqrt{a^2+b^2}$



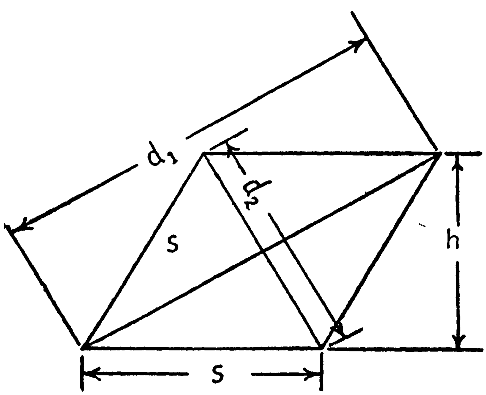
Square

$P=4s$
$A=s^2$
$d=s\sqrt{2}$
$s=\frac{1}{2}d\sqrt{2}$



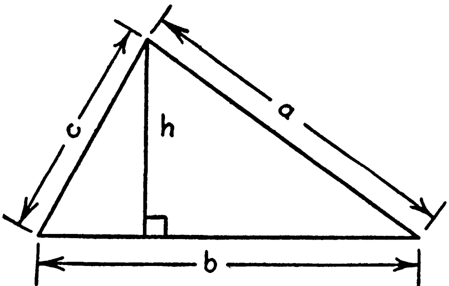
Parallelogram

$P=2(a+b)$
$A=ah$



Rhombus

$P=4s$
$A=sh$
$A=\frac{1}{2}d_1d_2$



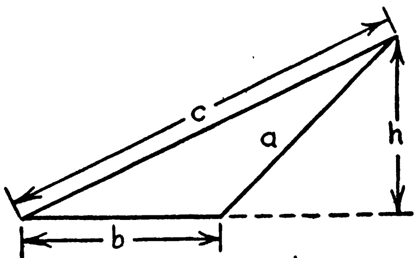
Acute Triangle

$P = a + b + c$

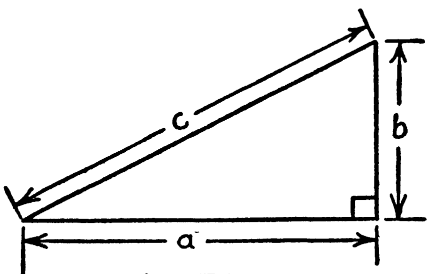
$A = \frac{1}{2}bh$

$A = \sqrt{s(s-a)(s-b)(s-c)}$

where s = semiperimeter



Obtuse Triangle



Right Triangle

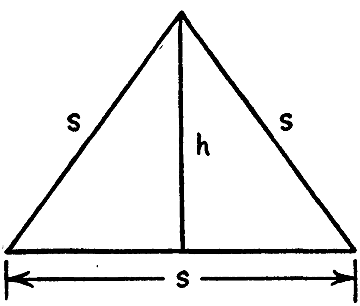
$P = a + b + c$

$A = \frac{1}{2}ab$

$c = \sqrt{a^2 + b^2}$

$a = \sqrt{c^2 - b^2}$

$b = \sqrt{c^2 - a^2}$

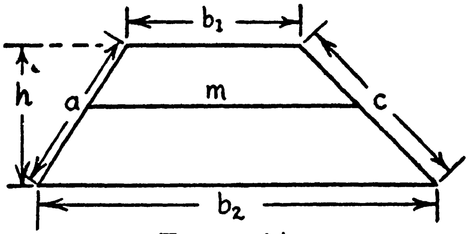


Equilateral Triangle

$P = 3s$

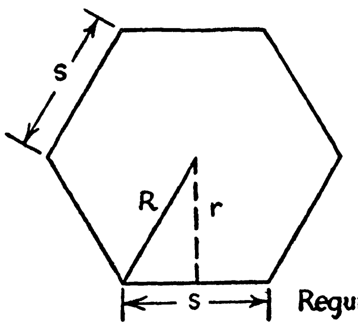
$A = \frac{s^2}{4}\sqrt{3}$

$h = \frac{s}{2}\sqrt{3}$



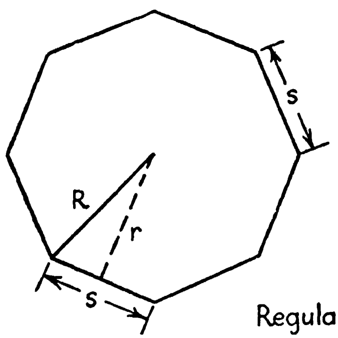
Trapezoid

$P=a+c+b_1+b_2$
$A=\frac{1}{2}h(b_1+b_2)$
$m=\frac{1}{2}(b_1+b_2)$
$A=\frac{1}{2}mh$



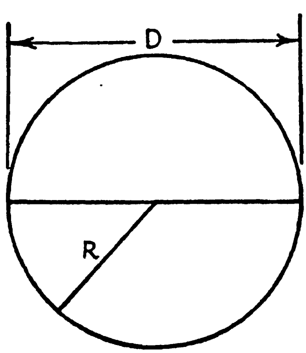
Regular Hexagon

$P=6s$
$R=s$
$r=\frac{s}{2}\sqrt{3}$
$A=\frac{3s^2}{2}\sqrt{3}$



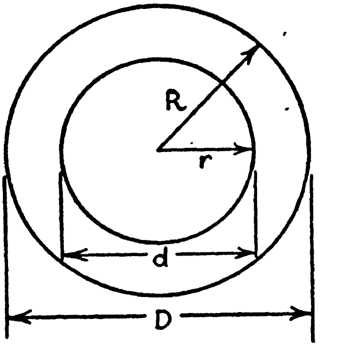
Regular n-gon

$P=ns$
$A=\frac{1}{2}Pr=\frac{1}{2}nsr$
$A=\frac{1}{2}(ns)\sqrt{R^2-\frac{s^2}{4}}$
$r=\sqrt{R^2-\frac{s^2}{4}}$



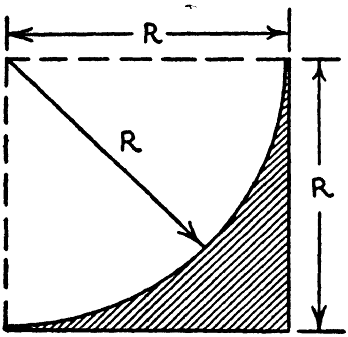
Circle

$C=2\pi R=\pi D$
$A=\pi R^2=\frac{\pi D^2}{4}=.7854D^2$



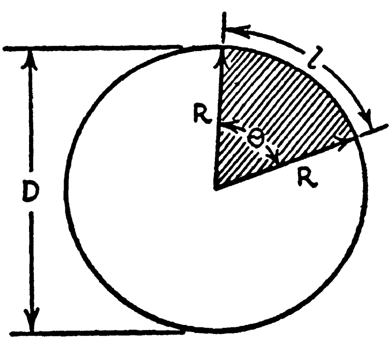
Ring

$$A=\pi(R^2-r^2)$$
$$A=\pi(R+r)\ (R-r)$$
$$A=\frac{\pi}{4}(D+d)\ (D-d)$$



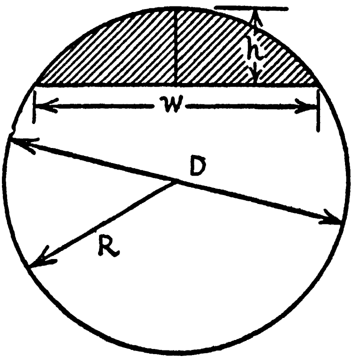
Fillet

$$P=2R+\frac{\pi R}{2}$$
$$A=R^2-\frac{\pi R^2}{4}$$



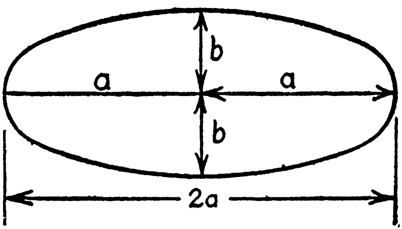
Sector

$$l\ (\text{arc})=\frac{\pi R\theta}{180}=\frac{\pi D\theta}{360}$$
$$A=\frac{\theta}{360}(\pi R^2)$$
$$A=\frac{\theta}{360}\left(\frac{\pi D^2}{4}\right)$$



Segment

$$h = R - \sqrt{R^2 - (\tfrac{1}{2}w)^2}$$
$$h = \tfrac{1}{2}(D - \sqrt{D^2 - w^2})$$
$$A = \frac{h(\tfrac{4}{3}w^2 + h^2)}{2w}$$

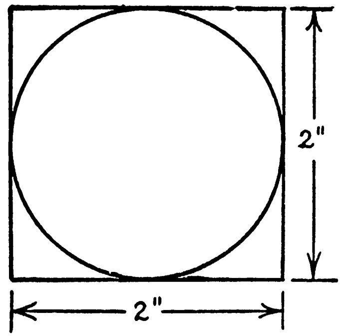


Ellipse

$P = \pi\sqrt{2(a^2 + b^2)}$ (approximately)
$A = \pi ab$

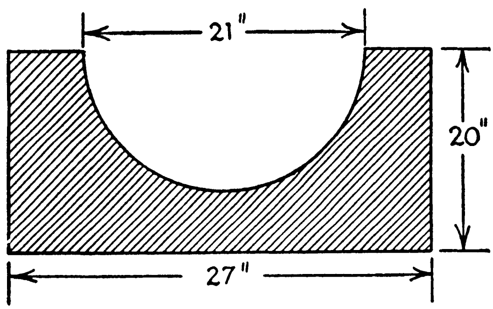
Exercise 65.

1. A circular-shaped rod is to be milled from a piece of square stock 2″ on a side. Find the area of the cross section of the finished rod.



2. Find the side of a square equivalent to a circle one foot in diameter.

3. Find the cross-sectional area of the semi-circular wooden trough shown in the diagram.



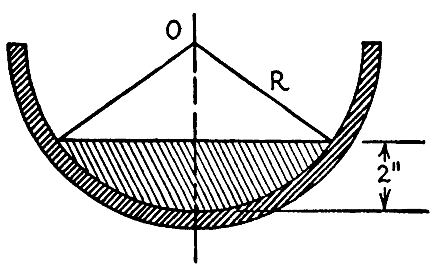
4. If the diameter of a circle is 40″, what is the area of a segment whose arc is 120°?

5. The diameters of two circles are 4″ and 6″, respectively. Find (a) the ratio of their circumferences; (b) the ratio of their areas.

6. What per cent of any circle is wasted if the largest possible hexagon is cut out of it?

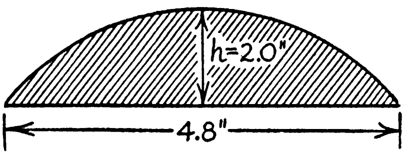
7. Find the cross-sectional area of the metal in a hollow shaft measuring 3½″ outside diameter and 2″ inside diameter.

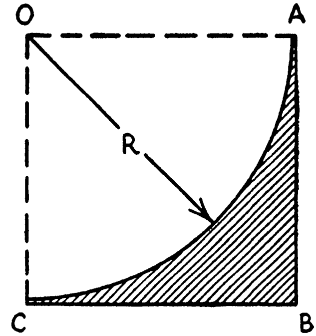
8. The inside diameter of a horizontal pipe is 10½″. Water is standing to a depth of 2″. Find the cross-sectional area of water in the pipe.



9. A wooden cylinder with a circular cross section is to be planed down to have the largest possible square cross section. If the diameter of the original rod is 3½″, what per cent is wasted in shavings?

10. A metal stamping has the shape of a circular segment with a span of 4.8 in. and a height of 2.0″. Find its area.



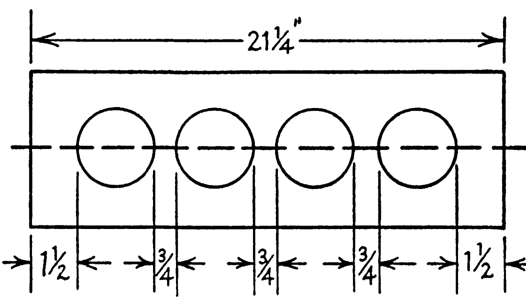


11. Find the area of the shaded fillet shown, where the radius of the fillet is 4″, and OABC is a square.

12. Find the area of a segment having for its chord a side of a regular inscribed hexagon, if the radius of the circle is 10½″.

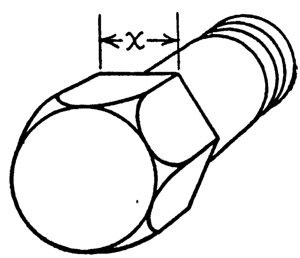
13. The end section of a gasoline tank is an ellipse with a major axis of 28″ and a minor axis of 22″. Find the area of this cross section of the tank.

14. The gasket of an auto engine is as shown; find the area of the metal wasted when the holes are punched out.



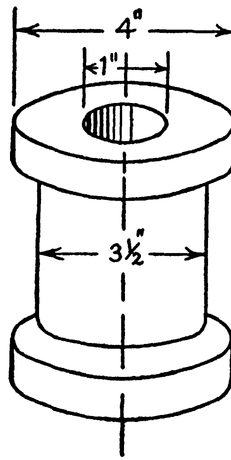
15. In a steam engine having a piston 18″ in diameter, the pressure upon the piston is 84 lb. per sq. in. Find the total pressure on the piston.

16. If $x = 1½''$, what is the diameter of the round stock needed to make the hexagonal head of this machine stud?



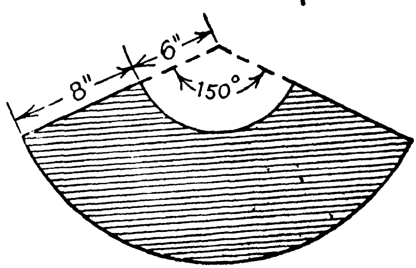
17. If a $3\frac{1}{2}''$ circular piece of rubber is cut from a sheet that is $4''$ square, what per cent of the material is left?

18. An $8''$-pipe supplies only $\frac{2}{3}$ the amount of water required in a given time. Assuming that the flow is proportional to the cross-sectional area, find the diameter of the smallest pipe that will supply the required amount, if the pipe sizes come only in diameters of whole numbers of inches.



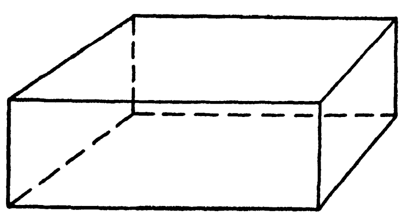
19. Find the area of the end of this spool before the hole is drilled; after the hole is drilled.

20. A parchment lampshade is to be made from a piece of material cut according to the pattern shown; find the area of the parchment required.

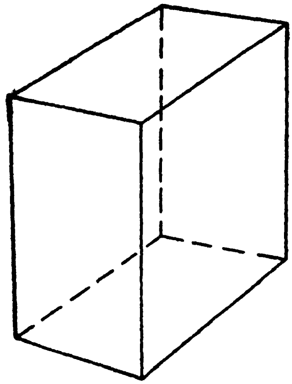


18. MEASUREMENT OF SOLID FIGURES

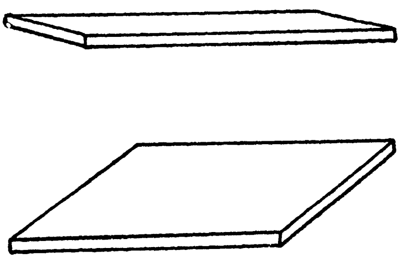
Rectangular Solids. A *rectangular solid* is a solid figure with 6 *faces* and 12 *edges;* all of the faces are rectangles, being, in fact, three pairs of congruent rectangles, and all the edges are perpendicular to the faces which they meet. The six faces are usually referred to as the "top and bottom," or the *bases;* as the "front and back,"

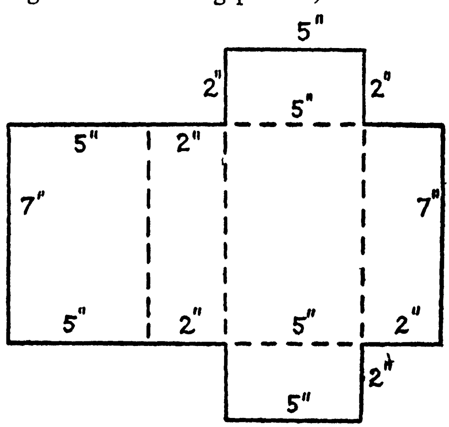


and as the "two ends," the last four constituting the *lateral* faces. Such figures are also known as rectangular *prisms,* but there are other types of prisms, too, as will be seen shortly. The three different edges are often known as the *length, width* and *height* (or thickness), although it is sometimes better to refer to the perpendicular distance between the bases as the *altitude* instead of the height or thickness.



Rectangular-solid shapes are met in shop practice in various proportions. Thus metal bars, planks, and joists are usually rectangular solids with one of the three "dimensions" much greater than the other two, as suggested in the diagram; thus we speak of a "two by four" which is 8 ft. long, meaning a joist 2 in.×4 in.×8 ft. Or again, in machine-shop work, frequent use is made of flat, rectangular *plates* and *sheets* of metal; if rectangular "pieces" are cut out, they are still rectangular solids, no matter how thin. A rectangular solid may be "developed" according to the following pattern, which will help to understand its geometric properties. In other words, by cutting along the solid lines, and folding along the dotted lines, a rectangular solid would be formed.





Area and Volume of a Rectangular Solid. The *lateral area* of a rectangular solid with dimensions l, w, and h is equal to $2lh+2wh$,

or, $L.A.=2h(l+w)$,

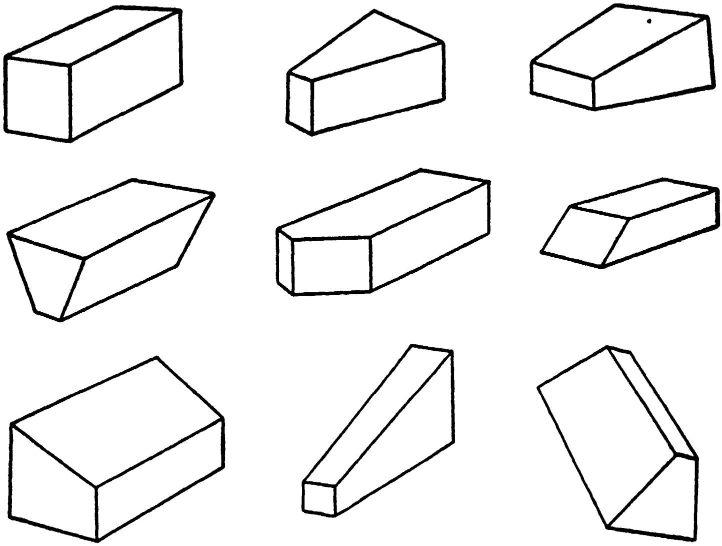
or, $L.A.=ph$, where $p=$ the perimeter of the base.

The *total area* is the lateral area plus the area of the two bases, or

$$T.A.=2lh+2wh+2wl.$$

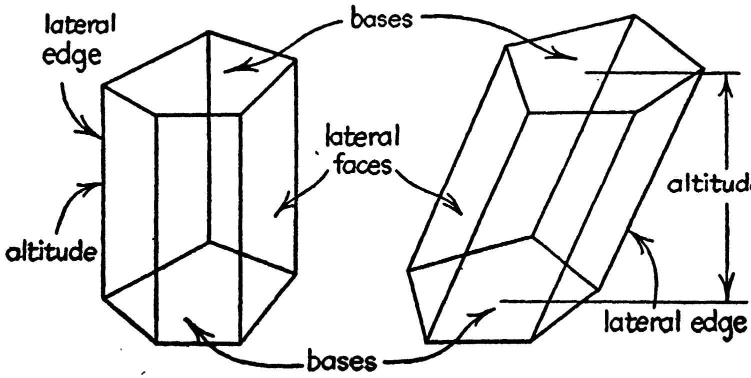
The *volume* of a rectangular solid, which has already been discussed, equals the product of its three dimensions; thus $V = lwh$,

or $V = Ah$, where A is the area of either base.

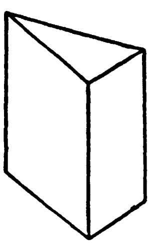


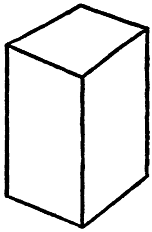
TYPICAL SHAPES OF FIRECLAY AND SILICA BRICK
(for high temperature work in the construction of ovens, furnaces and kilns.)

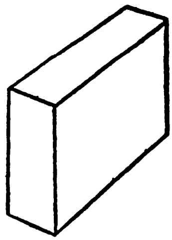
Prisms. Solids bounded partly by a series of flat surfaces (faces) whose intersections are parallel lines, and partly by a pair of additional parallel surfaces (bases) each of which intersects the series of lateral faces are called *prisms*. Such solids may be *right prisms* or *oblique prisms*, according as their *lateral edges* are perpendicular to the *bases* or not. The lateral

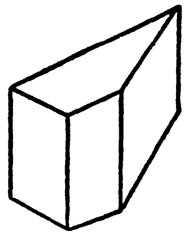


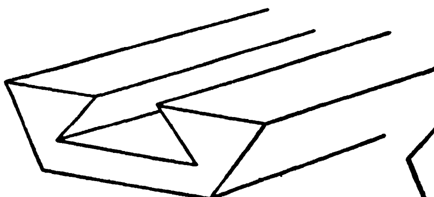
faces of a prism are always parallelograms; they may or may not be congruent, and they may or may not be rectangles. But their lines of intersection are always parallel, whether oblique or perpendicular to the bases. The bases are congruent polygons, parallel to one another, and each base intersects all the lateral faces. These bases may be triangles, squares,

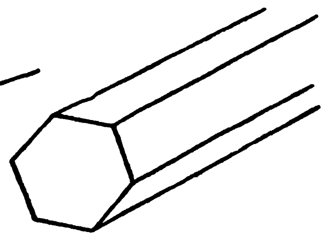

Triangular Prism

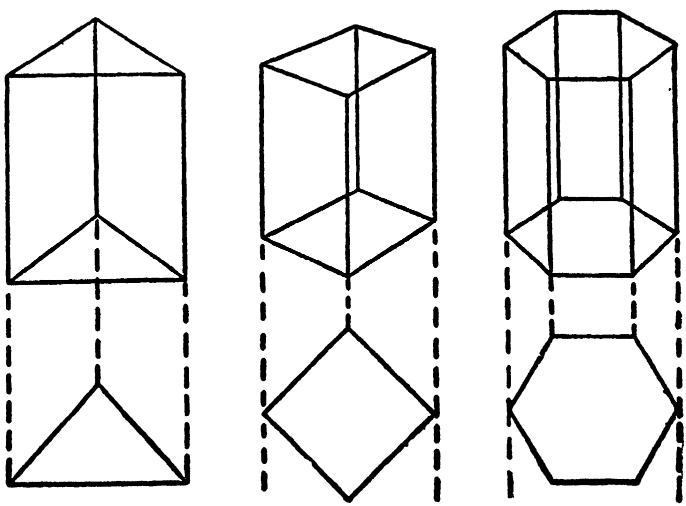

Square Prism


Rectangular Prism


Trapezoidal Prism

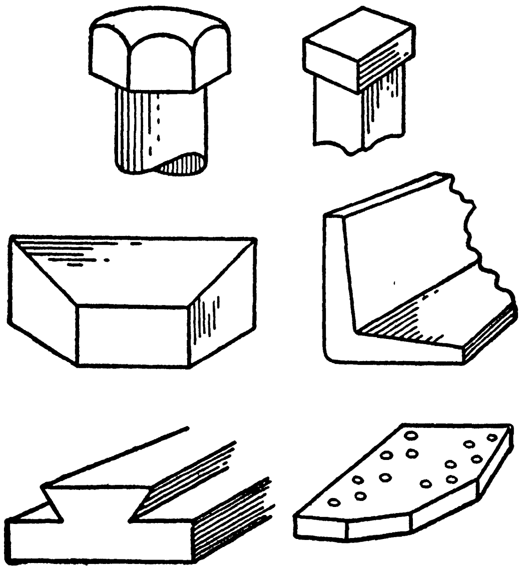

Grooved Prism


Hexagonal Prism

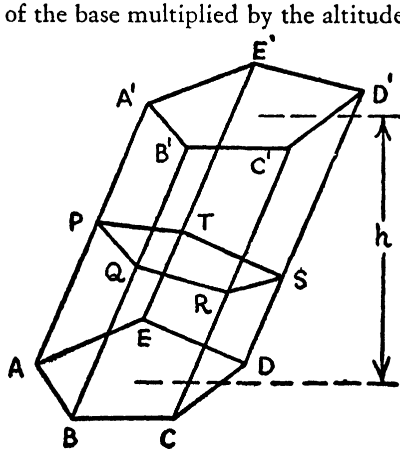


hexagons, etc. The *altitude* of any prism is the perpendicular distance between the two bases. Thus in a right prism the altitude is equal in length to any lateral edge, since all the lateral edges are perpendicular to the bases; in an oblique prism the altitude is less than the length of a lateral edge. In both types of prisms, however, all the lateral edges are equal in length.

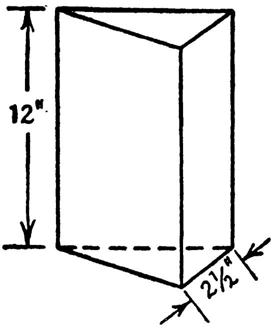
A *regular prism* is any right prism whose bases are regular polygons. In such prisms every non-lateral edge, i.e., base edge, equals every other non-lateral edge. In connection with mechanical work, many parts, such as strips, plates, beams, bars, sheets, angle irons, and the like are in the shape of regular prisms, or partly so, at any rate.



The *lateral area* of any regular prism equals the perimeter of the base times the altitude; the *total area* then equals the lateral area plus twice the area of the base. The *volume* of any regular prism equals the area of the base multiplied by the altitude. For specific types of prisms see the diagram above, or the summary of formulas near the end of this section. If, on the other hand, the figure is an oblique prism, its lateral area can be found by computing the area of each face separately, or by multiplying the perimeter of a right section by the length of a lateral edge; or *L.A.*= (PQ+QR+RST+· · ·)×AA′. The volume is still equal to the area of (ABCDE)×*h*.



EXAMPLE 1: Find the volume of a right prism whose base is an equilateral triangle 2½″ on a side and whose altitude is 12″.



SOLUTION:

$$V = Bh;\ B = \frac{s^2}{4}\sqrt{3}$$

$$B = \left(\frac{1}{4}\right)\left(\frac{25}{4}\right)\sqrt{3} = \frac{25}{16}(1.73) = 2.70$$

$V = Bh = (2.7)(12) = 32.4$ cu. in., *Ans.*

EXAMPLE 2: What is the lateral area of a regular hexagonal prism, ¾" on a lateral edge, and 3" on each edge of the base?

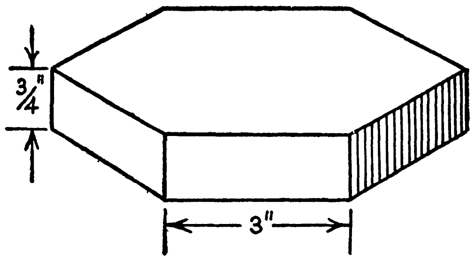
SOLUTION:

$$L.A. = (6)(3)\left(\frac{3}{4}\right) = 13.50$$

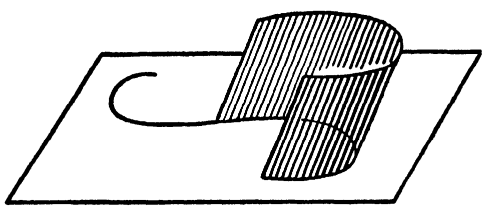
Area of base=

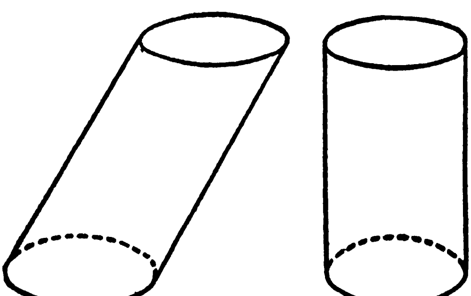
$$\left(6\right)\left(\frac{3^2}{4}\right)\sqrt{3} = 23.36$$

$T.A. = 13.50 + (2)(23.36)$
$= 60.2$ sq. in., *Ans.*

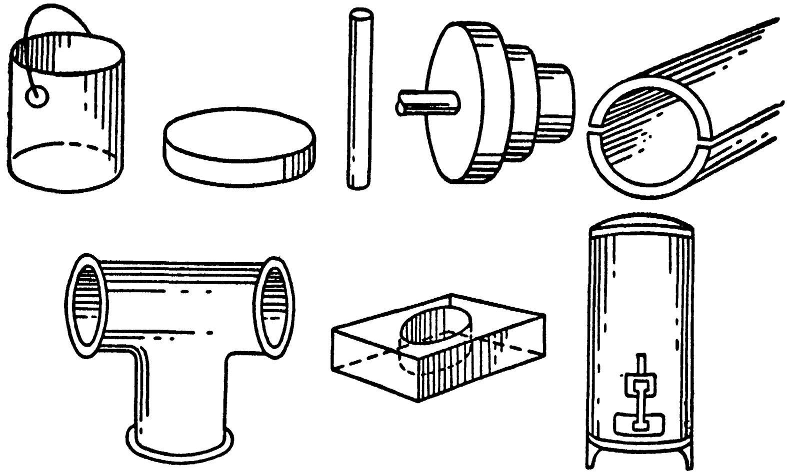


Cylinders. A *cylindrical surface* is one that has been generated by a straight line moving in such a way that it follows a fixed curve in a plane, while maintaining a constant angle to the plane. If the fixed curve (directrix) whose path is followed is a circle, the surface generated is a circularly cylindrical surface; if in addition the "generating line" remains perpendicular to the plane, it is a *right* circular cylindrical surface. A solid formed by such a surface *and* two parallel planes intersecting the cylindrical surface is called an *oblique circular cylinder* or a *right circular cylinder,* respectively. An approximate illustration of an oblique elliptical cylinder would be the shape of some steamship funnels.



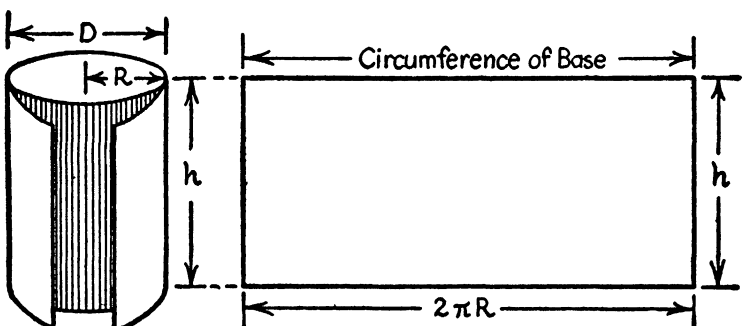


Right Circular Cylinders. Many common objects are in the form of a right circular cylinder: spools, drums, tubes, pipes, discs, pulleys, wheels, containers, cans, etc. In a right circular cylinder, the bases are equal circles, there are no "lateral edges," and the altitude is the perpendicular distance between the two bases. If the altitude of a right



circular cylinder is relatively small compared to its diameter, it is usually called a *disc;* if the altitude is considerably longer than its diameter, it is called a *rod* or a *tube,* according to whether it is solid or hollow.

The lateral area of a right circular cylinder will be seen from the fol-



lowing diagram to be given by

$$L.A. = 2\pi Rh = \pi Dh;$$

and the total area is

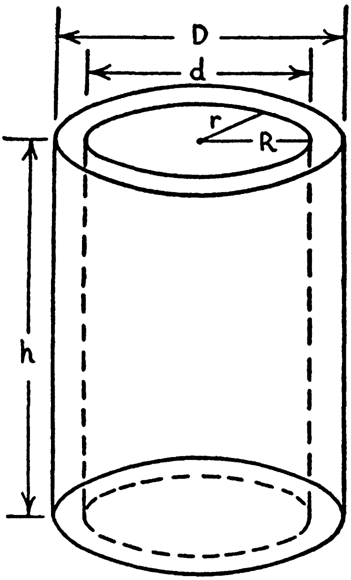
$$T.A. = 2\pi Rh + 2\pi R^2$$

$$\text{or, } T.A. = 2\pi R(h+R)$$

$$= \pi D\left(h + \frac{D}{2}\right)$$

The volume of a right circular cylinder, like that of a right prism, equals the area of the base multiplied by the altitude; or

$$V=\pi R^2h=\frac{\pi D^2}{4}h.$$



Hollow Cylinders. For such figures the following formulas will be found useful:

$$L.A.\ (inside)=2\pi rh$$
$$L.A.\ (outside)=2\pi Rh$$
$$End\ area\ (one)=\pi(R^2-r^2)$$
$$V=\pi h(R^2-r^2)$$
$$=\frac{\pi h}{4}(D^2-d^2)$$

Fillet. The volume of a fillet is given by the product of its cross-sectional area multiplied by its length (or height).

EXAMPLE 1: Find the contents in gallons of a cylindrical tank 7 ft. in diameter and 14 ft. high, if filled to a level 9½ ft. above the bottom.

SOLUTION: $V=\pi R^2h$

$$=\left(\frac{22}{7}\right)\left(\frac{49}{4}\right)\left(\frac{19}{2}\right)=365.75 \text{ cu. ft.}$$

1 cu. ft.=7½ gal.

365.75×7½=2743 gal., *Ans.*

EXAMPLE 2: What is the area of the inside surface of an open cylindrical tank measuring 3½ ft. in height and 1½ ft. in diameter?

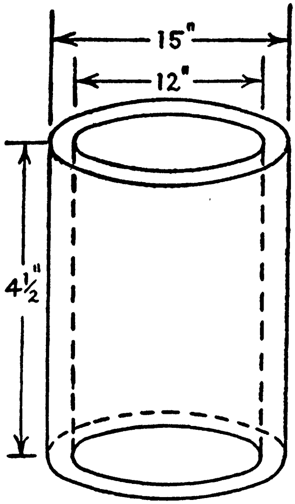
SOLUTION: $L.A.=2\pi Rh$

$$=2\left(\frac{22}{7}\right)\left(\frac{3}{4}\right)\left(\frac{7}{2}\right)=16.5 \text{ sq. ft.}$$

$B=\pi R^2h$

$$=\left(\frac{22}{7}\right)\left(\frac{9}{16}\right)\left(\frac{7}{2}\right)=6.2 \text{ sq. ft.}$$

Total area=22.7 sq. ft., *Ans.*



EXAMPLE 3: Find the weight of a section of cylindrical metal casting 4½ ft. long, with an inside diameter of 12″ and an outside diameter of 15″, if the material weighs 0.24 lb. per cu. in.

$$V=\frac{\pi h}{4}(D^2-d^2)$$

$$V=\left(\frac{22}{7}\right)\left(\frac{9}{2}\right)\left(\frac{12}{4}\right)(225-144)$$

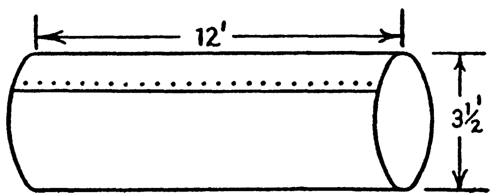
$V=3435$ cu. in.

Weight$=3435\times0.24=824$ lb., *Ans.*

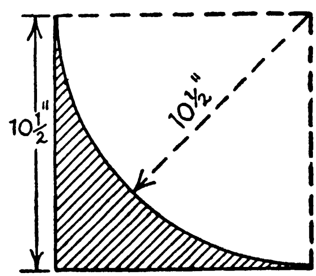
Exercise 66.

(*For reference:* 1 gal.=231 cu. in.; 1 cu. ft.= 7½ gal.; 1 cu. ft. water=62½ lb.)

1. What is the weight of a rectangular steel bar 48″×4″×1½″, if the metal weighs .25 lb. per cu. in.?
2. Allowing 4″ for overlapping at the seam along its entire length, how many sq. ft. of galvanized iron are required for this section of metal pipe?

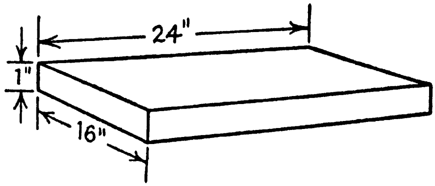


3. A fillet with the cross section shown is 14 ft. in length. How many cu. ft. of material are there in 20 such pieces?

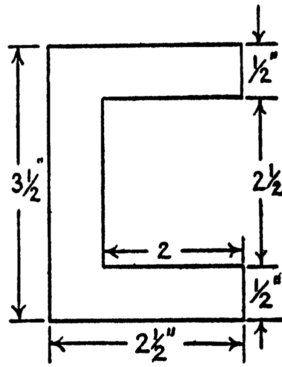


4. Brass discs ½″ thick are to be machined from 4″ diameter stock. (a) How many cu. in. of metal are there in each disc? (b) What is the weight of each disc if brass weighs .31 lb. per cu. in.?
5. An irregular piece of metal is placed in a cylindrical vessel 3 in. in diameter and partly full of water. If the water rises 3½ in., what is the volume of the piece of metal?

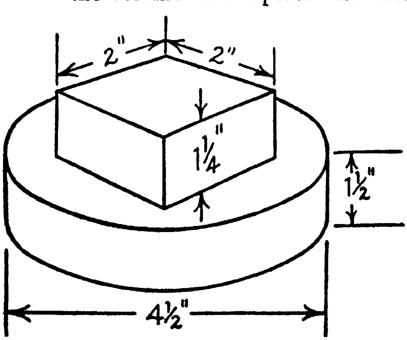
6. At 40¢ per lb., what is the cost of copper required to make 24 open pans with the dimensions shown, if sheet copper weighs .8 lb. to the square foot, and 10% is allowed for lapping and waste?

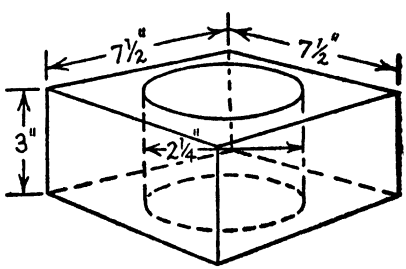


7. Find the weight of a cast-iron cylinder 2.80 in. in diameter and 2 ft. long, if cast iron weighs 448 lb. per cu. ft.
8. Hollow metal bars in the shape of a regular hexagon in 12 ft. lengths are to be plated with a special chromium finish. If the cross section of the bar is a regular hexagon ¾″ on a side, find the total surface to be plated on 100 such bars.
9. Laminations for an electrical machine are made of pieces of sheet iron with the shape and dimensions shown. What is the weight of 1000 such pieces, if the iron used weighs .48 lb. per sq. ft.?

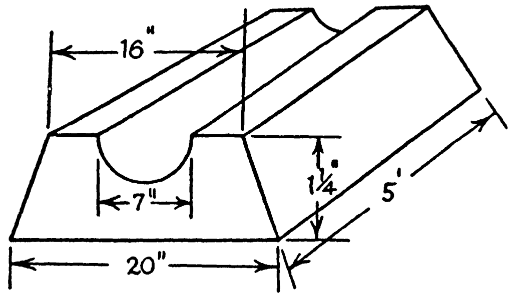


10. A cylindrical hot water boiler has a diameter of 14″ and stands 4′ 8″ high. How many gallons will it hold when full?
11. A rectangular block of gold metal measures 10″×8″×2″. If it is hammered and rolled into sheets 8″ square and .02″ thick, how many such sheets will be obtained?
12. Find the volume of a cylindrical bronze ring having an inside diameter of 4⅜″ and an outside diameter of 6¼″, if it is ¾″ thick.
13. Find the volume of the solid iron plug with dimensions as shown.
14. A circular hole is drilled through the metal block as shown; what is the volume of the piece after the hole has been drilled?



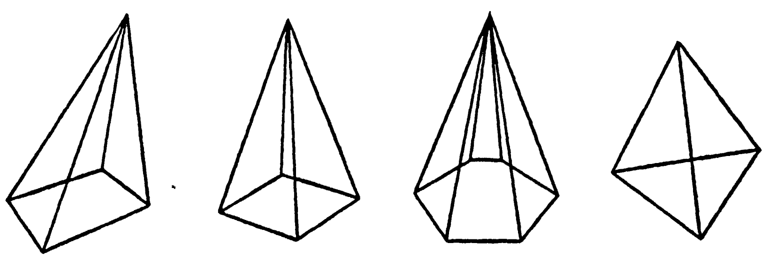


15. A rod of copper 14 ft. long and 2 sq. in. in cross section is melted and formed into a wire ⅛ in. in diameter. Find the length of the wire.
16. Concrete blocks like the one shown in the sketch are made in molds. If the sections are 5 ft. long and 7¼ in. high, how many cu. yd. of concrete are required for 48 such sections?



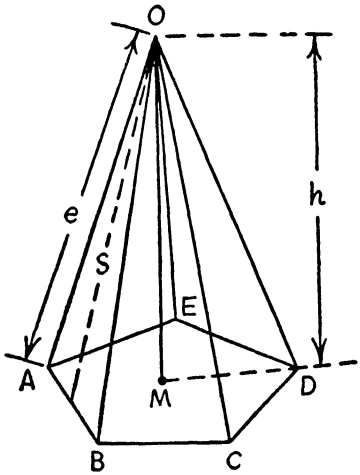
17. If 2 gallons of water are poured into a cylindrical jar 7 inches in diameter, how high will the level of the water rise?
18. Sections of reinforced concrete water main are hollow cylinders 12 ft. long and 4″ thick. If the inside diameter is 48″, and the concrete weighs 105 lb. per cubic foot, what is the weight of one such section?
19. A cylindrical paint can 14 in. high holds exactly 2 gallons. What is the diameter of the can?
20. In a building there are 2100 ft. of steam piping with a 12″ outside diameter. How many sq. ft. of pipe surface does this represent?
21. If a bar of metal 2″ in diameter weighs 10.2 lb. per foot of length, what is the weight per running foot of a bar 3″ square of the same material?
22. A tunnel whose cross section is a semicircle 21 ft. high is ¼ mile long. How many cu. yd. of earth were removed when it was excavated?

Pyramids. Any solid figure, one of whose faces is a polygon and whose other faces are all triangles having a common vertex, is a *pyramid.*



Oblique Pyramid | Regular Square Pyramid | Regular Hexagonal Pyramid | Regular Tetrahedron

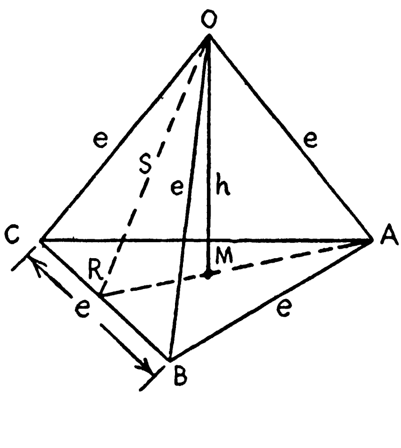
The *altitude* of any pyramid is the perpendicular distance from the



vertex (O) to the plane of the base. If the base is a regular polygon, and the altitude of the pyramid passes through the center of the base, the solid is a *regular pyramid.* In such a regular pyramid, all the lateral faces are obviously equal isosceles triangles; the altitude of any lateral face is known as the *slant height* of the pyramid, and is always greater than the altitude of the pyramid.

The lateral surface of a regular pyramid is given by $L.A. = \frac{1}{2}Ps$, where P=perimeter of the base and s=slant height. The total area, of course, is the lateral area plus the area of the base. The volume of any pyramid, whether regular or not, equals ⅓ the area of its base multiplied by its altitude.

Regular Tetrahedron. A special case of a regular pyramid is the case of a regular triangular pyramid in which not only the base, but each of the three lateral faces, are all equilateral triangles. Such a figure is called a *regular tetrahedron,* since all four faces are identical. If each of the six equal edges is denoted by e, then careful study of the figure will show that the following relations hold:



(1) $OR = s = \frac{e}{2}\sqrt{3}$

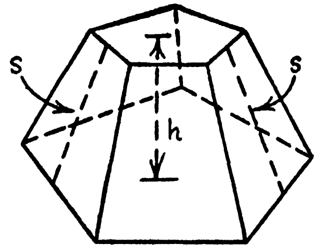
(2) $AR = OR = \frac{e}{2}\sqrt{3}$

(3) $AM = \frac{2}{3}(AR) = \frac{e}{3}\sqrt{3}$

(4) $OM = h = \sqrt{(OA)^2 - (AM)^2}$

$$= \sqrt{e^2 - \left(\frac{e\sqrt{3}}{3}\right)^2} = \frac{e}{3}\sqrt{6}$$

Frustum of a Pyramid. If a portion of a pyramid including the vertex is cut off by a plane parallel to the base, the part left is called the *frustum*

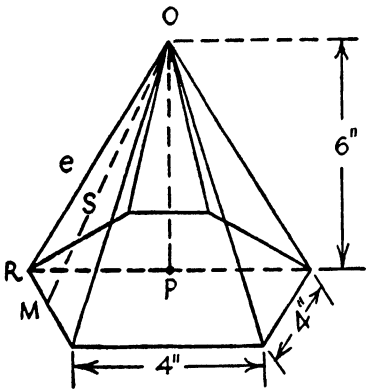


of the pyramid. Thus in the frustum of a regular pyramid each lateral face is an isosceles trapezoid; all these lateral faces are equal to one another, and the slant height of the frustum is the altitude of the trapezoid. The altitude of the frustum is the perpendicular distance between the two bases, which are similar regular polygons.

The lateral area of the frustum of a regular pyramid equals half the sum of the perimeters of the bases multiplied by the slant height, or

$$L.A.=\tfrac{1}{2}(P+p)s.$$

The volume of a regular frustrum is given by $V=\tfrac{1}{3}h(B+b+\sqrt{Bb})$, where b and B are the areas of the upper and lower bases, respectively.



EXAMPLE 1: Find the lateral area and volume of a regular hexagonal pyramid whose altitude is 6″, if the sides of the base are 4″ each.

SOLUTION:

$$(OR)^2=(OP)^2+(PR)^2$$
$$e^2=4^2+6^2=16+36=52$$
$$(OM)^2=(RM)^2+(OR)^2$$
$$s^2=4+52=56$$
$$s=\sqrt{56}=7.5$$

$L.A.=\tfrac{1}{2}ps=\tfrac{1}{2}(24)(7.5)=90.0+$ sq. in., *Ans.*

$V=\tfrac{1}{3}Bh=\tfrac{1}{3}[(6)(4\sqrt{3})(6)]=83.0+$cu. in., *Ans.*

EXAMPLE 2: Find the total area and volume of the frustum of a square pyramid with dimensions as shown.

SOLUTION: $s^2=5^2+2^2=25+4=29$

$s=\sqrt{29}=5.4$

$L.A.=\tfrac{1}{2}s(p_1+p_2)$

$=\tfrac{1}{2}(5.4)(24+40)$

$=172.8$ sq. in.

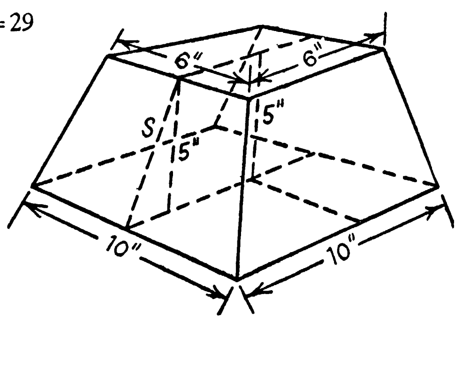
$T.A.=172.8+36+100$

$=308.8$ sq. in., *Ans.*

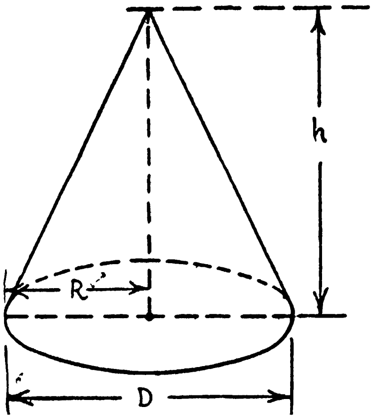
$V=\tfrac{1}{3}h(B_1+B_2+\sqrt{B_1B_2})$

$=\tfrac{1}{3}(5)(36+100+60)$

$=326.7$ cu. in., *Ans.*



Right Circular Cones. The properties of right circular cones are analogous to those of regular pyramids, of which, indeed, the cone can be considered as the limiting case as the number of sides is increased indefinitely. Hence we have:

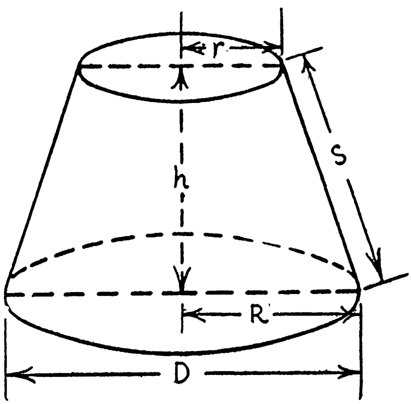


$$L.A.=\tfrac{1}{2}\,Cs=\tfrac{1}{2}(2\pi R)s=\pi Rs$$

$$T.A.=\pi R^2+\pi Rs=\pi R(R+s)$$

$$V=\tfrac{1}{3}Bh=\tfrac{1}{3}\pi R^2h=\frac{\pi D^2h}{12}$$

Similarly for the frustum of a regular cone:

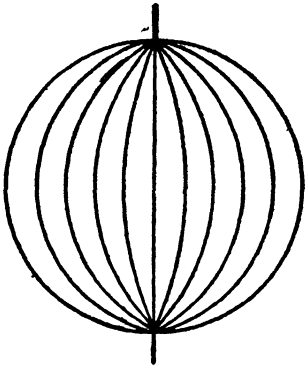


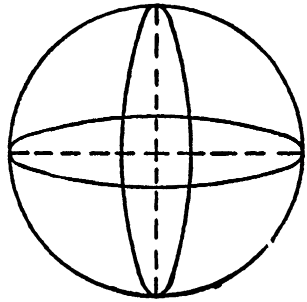
$$L.A.=\tfrac{1}{2}(C+c)s$$
$$=\tfrac{1}{2}(2\pi R+2\pi r)s$$
$$=\pi s(R+r).$$

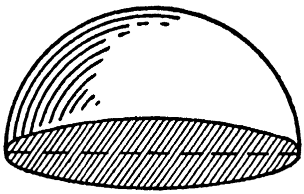
$$T.A.=\pi s(R+r)+\pi R^2+\pi r^2$$
$$=\pi[(R+r)s+R^2+r^2]$$

$$V=\tfrac{1}{3}h[\pi R^2+\pi r^2+\sqrt{\pi R^2+\pi r^2}]$$
$$=\tfrac{1}{3}\pi h[R^2+r^2+Rr]$$

Sphere. In practical use, the sphere or ball finds its commonest application where friction between moving parts is to be a minimum, as in ball bearings, or where the symmetry and "perfect beauty" of a sphere is attractive, as in ornamental parts. The sphere is a "closed" curved surface, every point on which is equally distant from a point within called the center. A sphere may also be thought of as the surface generated by revolving a semicircle about its diameter as an axis. A plane passed through the center of a sphere divides it into two equal *hemispheres*. The base of each hemisphere is a circle whose diameter is the same as that of the sphere; they are known as *great circles*.







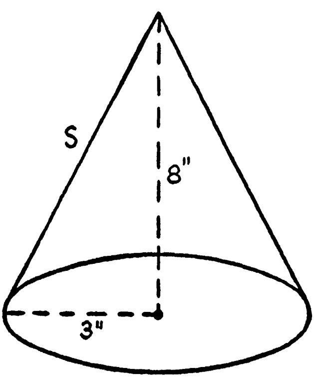
The surface of a sphere is found to be four times that of a great circle of the sphere, or

$$A=4\pi R^2=\pi D^2;$$

and the volume is given by

$$V=\tfrac{4}{3}\pi R^3=\frac{\pi D^3}{6}.$$

EXAMPLE 1: Find the total area and volume of a right circular cone if the altitude is 8″ and the diameter of the base is 6″.



SOLUTION:

$$s=\sqrt{8^2+3^2}=\sqrt{73}=8.6$$
$$L.A.=\pi Rs$$
$$=(\tfrac{22}{7})(3)(8.6)=81.1 \text{ sq. in.}$$
$$T.A.=L.A.+(\tfrac{22}{7})(9)$$
$$=81.1+28.3=109.4 \text{ sq. in., } Ans.$$
$$V=\pi R^2h=(\tfrac{22}{7})(9)(8)=$$
$$226.4 \text{ cu. in., } Ans.$$

EXAMPLE 2: Find the lateral area and volume of the frustum of a right circular cone whose altitude is 10″, and the diameters of whose bases are 8″ and 18″, respectively.

SOLUTION:

$s^2=10^2+5^2$

$s=\sqrt{100+25}=\sqrt{125}=11.2$

$L.A.=\pi s(R+r)$

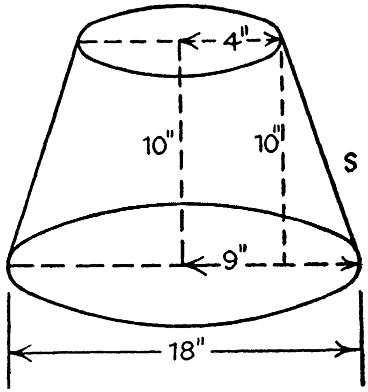
$=(^{22}\!/\!_7)(11.2)(9+4)=$

44.3 sq. in., *Ans.*

$V=\frac{1}{3}\pi h[R^2+r^2+Rr]$

$=(\frac{1}{3})(^{22}\!/\!_7)(10)[81+16+36]$

$=1393+$ cu. in., *Ans.*



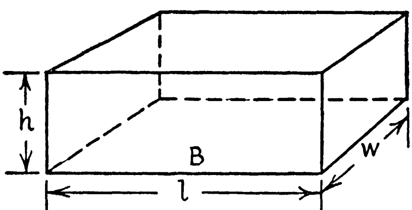
EXAMPLE 3: A hollow metal container is in the shape of a hemisphere whose diameter is 42 cm. What is its capacity?

SOLUTION: $V=\frac{4}{3}\pi R^3(\frac{1}{2})$

$=(\frac{4}{3})(^{22}\!/\!_7)(21^3)(\frac{1}{2})=19{,}404$ cu. cm., *Ans.*

SUMMARY OF FORMULAS

Areas and Volumes of Solid Figures

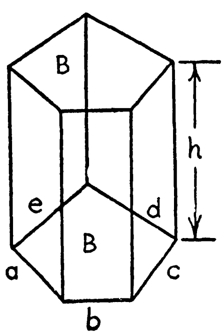


Rectangular Solid

$L.A.=ph=2h(l+w)$

$T.A.=2(lw+wh+wl)$

$V=lwh=Bh$

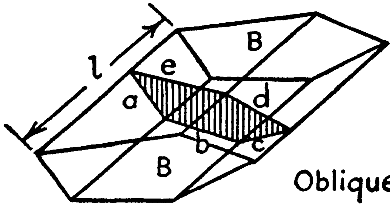


$L.A.=ph=h(a+b+c+\cdots)$

$T.A.=ph+2B$

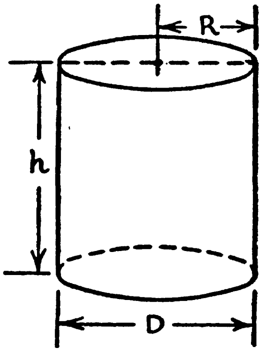
$V=Bh$

Right Prism



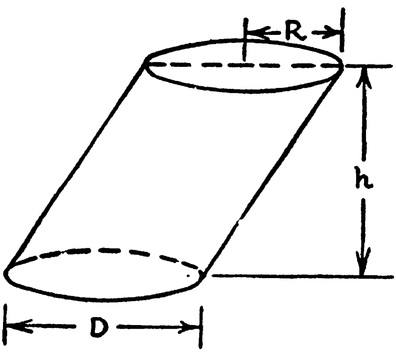
$L.A. = pl = h(a+b+c+\cdots)$
$T.A. = pl + 2B$
$V = Bh$

Oblique Prism



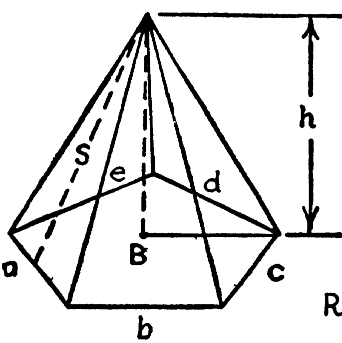
$L.A. = 2\pi Rh$
$T.A. = 2\pi Rh + 2\pi R^2$
$= 2\pi R(h+R)$
$V = \pi R^2 h$
$= \frac{\pi h}{4} D^2$

Right Circular Cylinder



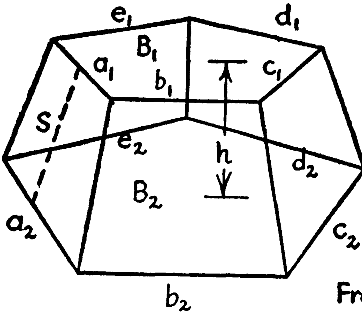
$V = \pi R^2 h$

Oblique Circular Cylinder



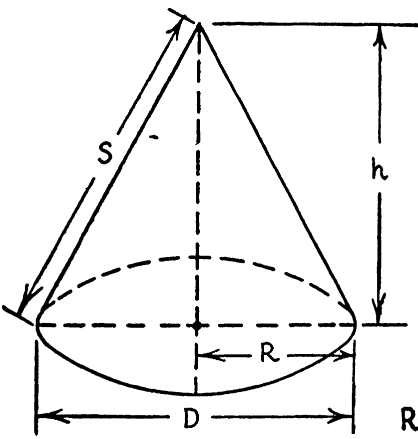
$L.A. = \frac{1}{2}ps = \frac{1}{2}s(a+b+c+\cdots)$
$T.A. = \frac{1}{2}ps + B$
$V = \frac{1}{3}Bh$

Regular Pyramid



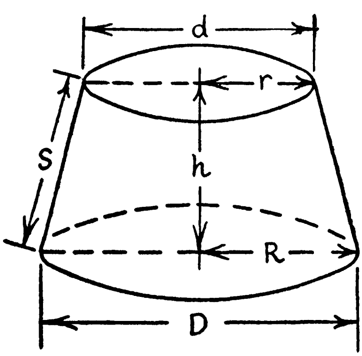
$$L.A. = \tfrac{1}{2}s(p_1+p_2)$$
$$= \tfrac{1}{2}s[(a_1+b_1+c_1+\cdots)$$
$$+(a_2+b_2+c_2+\ldots)]$$
$$T.A. = \tfrac{1}{2}s(p_1+p_2)+B_1+B_2$$
$$V = \tfrac{1}{3}h[B_1+B_2+\sqrt{B_1+B_2}]$$

Frustum of Regular Pyramid



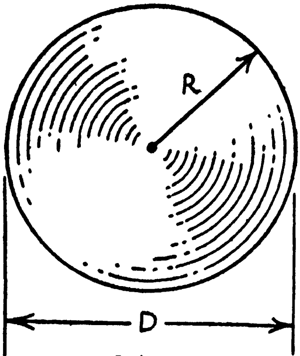
$$L.A. = \pi Rs$$
$$T.A. = \pi Rs+\pi R^2$$
$$= \pi R(s+R)$$
$$V = \tfrac{1}{3}\pi R^2 h$$
$$= \frac{\pi D^2 h}{12}$$
$$= .2618D^2h$$

Right Circular Cone



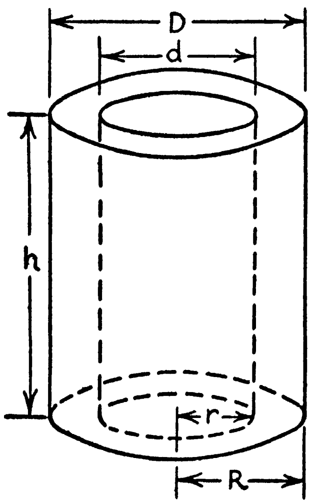
$$L.A. = \tfrac{1}{2}s(C+c)$$
$$= \tfrac{1}{2}s(2\pi R+2\pi r)$$
$$= \pi s(R+r)$$
$$T.A. = \pi s(R+r)+\pi R^2+\pi r^2$$
$$= \pi(R^2+r^2+Rs+rs)$$
$$V = \tfrac{1}{3}\pi h(R^2+r^2+Rr)$$
$$= .2618h(D^2+d^2+Dd)$$

Frustum of Right Circular Cone



Sphere

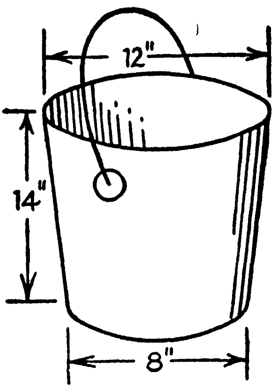
$$A = 4\pi R^2 = 12.566R^2 = \pi D^2$$
$$V = \tfrac{4}{3}\pi R^3 = 4.1888R^3$$
$$= \frac{\pi D^3}{6} = .5236D^3$$

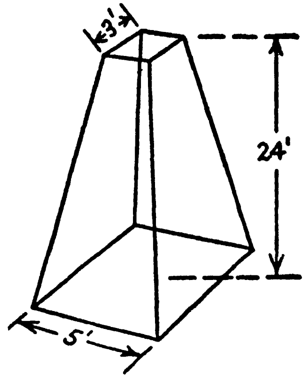


Hollow Cylinder

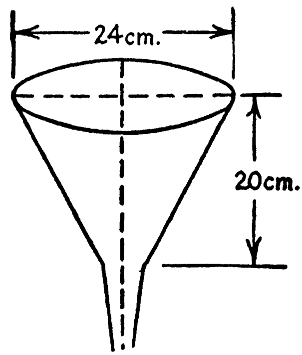
$$V = \pi h(R^2 - r^2)$$
$$= \frac{\pi h}{4}(D^2 - d^2)$$

Exercise 67.

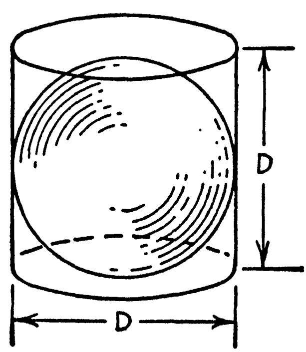




1. How many gallons can this bucket hold, if it has the form of a frustum of a right circular cone with dimensions as shown?
2. A concrete foundation pillar 24 ft. high is a frustum of a square pyramid whose bases are 3 ft. and 5 ft. on a side. How many cu. yd. of concrete are required for the pillar?



3. A funnel measures 24 cm. across the top, and 20 cm. from top to bottom, excluding the stem. Approximately how many cu. cm. of liquid will it hold when full?

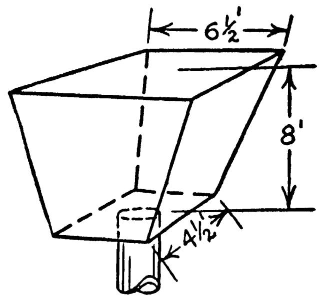


4. A sphere when immersed in a cylinder full of water just touches the sides and bottom, and is flush with the surface of the water remaining. What per cent of the water in the cylinder originally will the overflow be?

5. How many sq. in. of sheet metal will be needed to make a pail 12″ in diameter at the bottom and 14″ across at the top, with a slant height of 18″?

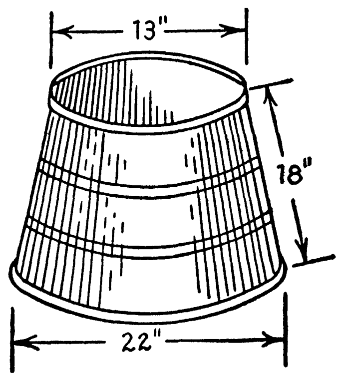
6. The altitude of a regular hexagonal pyramid is 17½″; the hexagonal base measures 4″ across the flats. Find the volume of the pyramid.

7. A feed hopper is in the form of the frustum of a square pyramid with dimensions shown. Find its cubic contents when filled to its maximum height of 8 ft.



8. A conical pile of sand measures 66 ft. around the circumference of its base. If the slope of the pile is 45°, how many cu. yd. of sand are there in the pile?

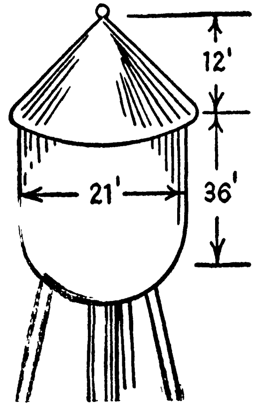
9. A parchment lamp shade has the shape of the frustum of a circular cone with the dimensions shown; find the amount of material required to make it up.



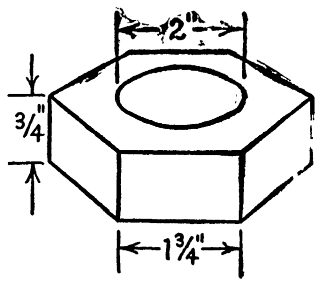
10. A cylindrical "capsule" with hemispherical ends has a diameter of 18″ and is 4½ feet in total length. Find its volume.

11. A regular triangular prism of glass measures 16.2 cm. in length; the side of each base is 1.8 cm. long. Find the lateral surface of the prism.
12. Find the volume of metal in a hollow sphere if the metal is 1½″ thick and the inside diameter equals 6″.
13. A cone pulley is a frustum of a cone with an altitude of 10½″; the diameters of the bases are 4″ and 6″, respectively. If a 1″ hole is drilled through the pulley, find the volume of the pulley.
14. Find the ratio between the surfaces of two spheres whose radii are 3″ and 6″.

15. A water tank in the shape of a cylinder with a hemispherical bottom and a conical top has the dimensions shown. What is its maximum capacity in gallons?

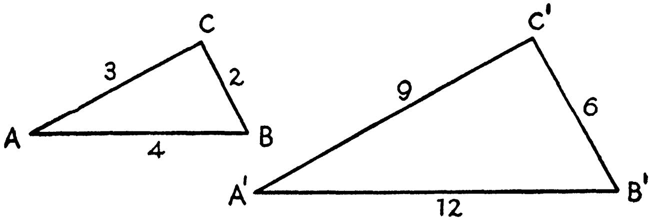


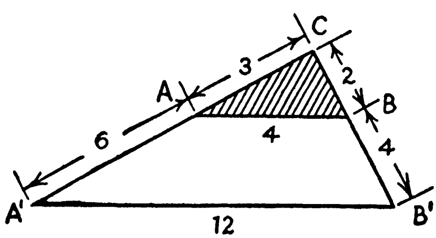
16. A 2″ hole is drilled through a hexagonal piece of stock ¾″ thick and 1¾″ across the flats. Find the volume of metal in the finished piece.



19. SIMILAR FIGURES

Similar Triangles. It was previously pointed out that any two (or more) triangles having the same shape, although differing in size, were said to be *similar triangles*. More specifically, having the "same shape" means





that the corresponding angles of the two triangles are respectively equal, and that their corresponding sides are in proportion. Thus: $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$; also, $\frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$.

In other words, in this particular instance, any pair of corresponding sides of the two triangles has the ratio 3:1, (or 1:3); thus:

$$\frac{AC}{A'C'} = 3/9 = 1/3; \quad \frac{AB}{A'B'} = 4/12 = 1/3;$$
$$\frac{B'C'}{BC} = 6/2 = 3:1; \quad \frac{A'C'}{AC} = 9/3 = 3:1; \text{ etc.}$$

As a matter of fact, *congruent* triangles may be regarded as a "special case" of similar triangles whose corresponding sides are in the ratio of 1:1.

Conditions for Similarity. Triangles are similar under any one of the following conditions:

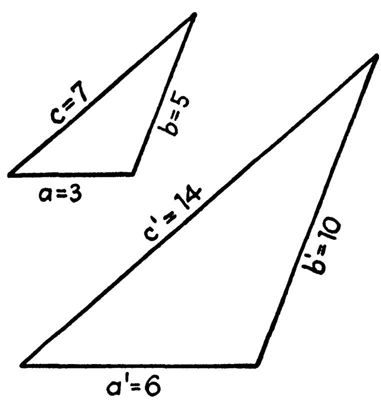
(1) if all three pairs of corresponding angles are equal;
(2) if any two pairs of corresponding angles are equal;
(3) if one pair of corresponding angles is equal, and the sides including them are in proportion;
(4) if all three pairs of corresponding sides are in proportion;
(5) if they are right triangles having one pair of acute angles equal.

Proportional Parts. As has just been seen, if two triangles are similar, their corresponding sides are in proportion; or

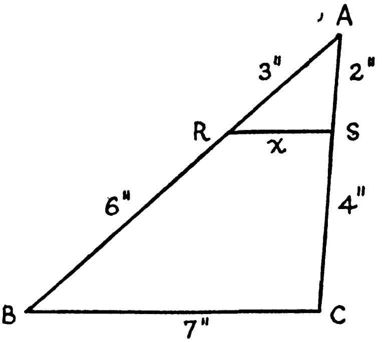
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$
$$3/6 = 5/10 = 7/14 = (1/2).$$

Or again; $\frac{a}{b} = \frac{a'}{b'}$; $\frac{a}{c} = \frac{a'}{c'}$; $\frac{b}{c} = \frac{b'}{c'}$

$$3/5 = 6/10; \quad 3/7 = 6/14; \quad 5/7 = 10/14.$$



Moreover, if a line (RS) is drawn parallel to any side (BC) of a triangle, it divides the other two sides into proportional parts, since the triangle cut off by the line is similar to the original triangle. Thus if RS divides AB in the ratio of 3:6 (or 1:2), it will also divide AC in the ratio of 1:2 (2:4). In other words,



$$\frac{AR}{RB}=\frac{AS}{SC}, \text{ or } 3/6=2/4.$$

Also, the length of RS can be determined if BC is known, for:

$$\frac{AR}{AB}=\frac{RS}{BC}, \text{ or } 3/9=\frac{x}{7};\ x=2⅓.$$

$$\text{Or again, } \frac{RS}{BC}=\frac{AS}{AC}, \text{ or } \frac{x}{7}=2/6;\ x=2⅓.$$

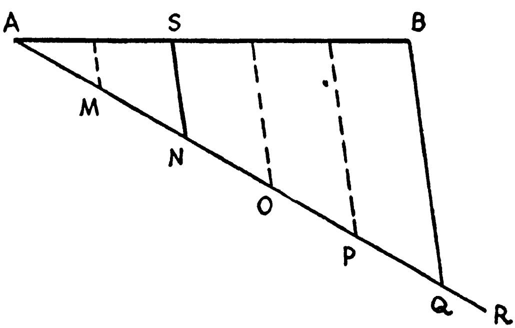
Furthermore, any segment is to the entire side as the corresponding segment is to the other side; i.e.,

$$\frac{AR}{AB}=\frac{AS}{SC}, \text{ or } \frac{3}{3+6}=\frac{2}{2+4};$$

$$\text{and } \frac{RB}{AB}=\frac{SC}{AC}, \text{ or } \frac{6}{3+6}=\frac{4}{2+4}.$$

Dividing a Line into a Given Ratio. A simple method for dividing a given line into any number of parts having any desired ratio is the following construction, based on similar triangles.

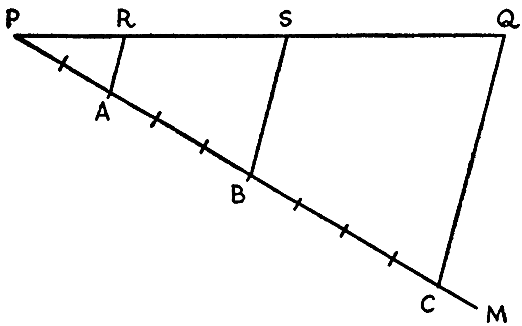
(a) Suppose that the line AB is to be divided into two parts having the ratio 2:3. At any convenient angle, draw an indefinite line through A. On AR step off five convenient equal segments, AM, MN, NO, OP, and PQ. Join Q with B, and through N construct NS || QB. The two required segments are AS and SB.



For, by the construction, △ANS is similar to (∼) △AQB; therefore

$$\frac{AN}{NQ}=\frac{AS}{SB}=2/3.$$

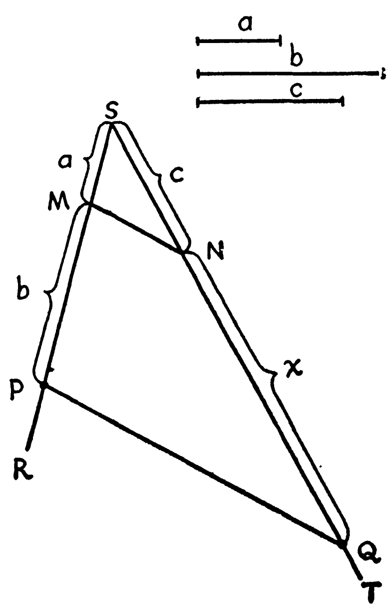
(b) Suppose that a given line PQ is to be divided into three parts having the ratio 2:3:4; a similar procedure is followed. An indefinite line PM is drawn at any convenient angle to PQ; 9 equal segments are then stepped off on PM, beginning at P and ending at C. Point C is joined with Q, and lines constructed through A and B, respectively, parallel to QC, and intersecting PQ in R and S. The required segments are PR, RS and SQ; for



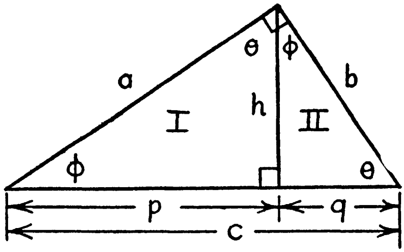
$$PA:AB:BC=PR:RS:SQ=2:3:4$$

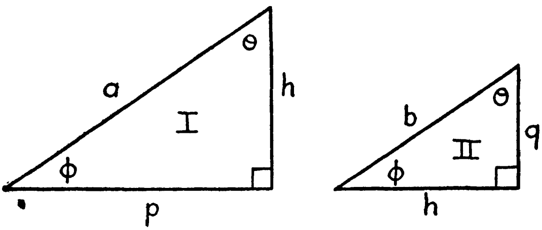
Constructing a Fourth Proportional. The same principle may be used to construct a fourth proportional to three given lines; for example, find the fourth proportional (x) to the given segments a, b and c. On any convenient angle RST with indefinite sides, lay off the given segments a, b and c as shown. Join the ends M and N of segments a and c; through P, the end of segment b, construct a line parallel to MN, intersecting ST in Q. Then NQ is the required fourth proportional x; for

$$\frac{SM}{MP}=\frac{SN}{NQ}, \text{ or } \frac{a}{b}=\frac{c}{x}.$$



The Altitude upon the Hypotenuse. A little consideration will show that if the altitude of a right triangle is drawn to the hypotenuse, the two triangles thus formed are not only similar to each other, but each is similar to the original triangle as well. As a result, the following relations hold; they are occasionally useful in computations and in deriving other properties of such figures:



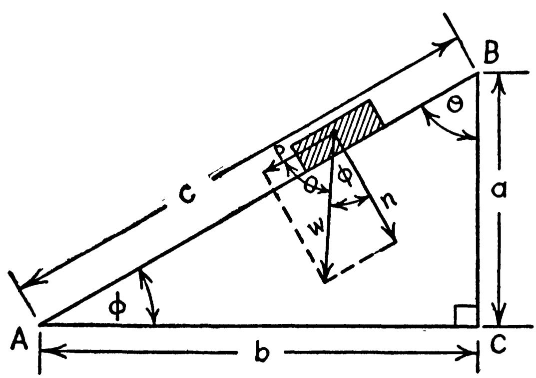


(1) $\frac{h}{p}=\frac{q}{h}$, or $h^2=pq$, $h=\sqrt{pq}$.

(2) $\frac{a}{p}=\frac{p+q}{a}$, or $a^2=p\ (p+q)$;
$a^2=pc$, or $a=\sqrt{pc}$.

(3) $\frac{b}{q}=\frac{p+q}{b}$, or $b^2=q\ (p+q)$;
$b^2=qc$, or $b=\sqrt{qc}$.

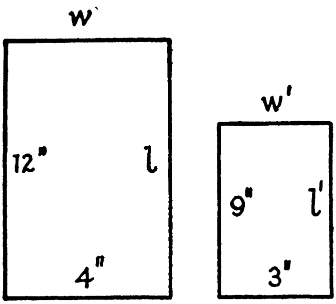
Practical Problem. A useful application of similar triangles occurs in connection with a study of forces acting on a body on an inclined plane. Thus the triangle of which w and p (the downward pull of the weight, and the pull along the inclined plane, respectively) are two sides of a triangle is similar to △ ABC; so is the triangle having w and n for two of its sides, where n is the "normal" (i.e., perpendicular) push upon the plane. Hence:



$\frac{p}{w}=\frac{a}{c}$, or $p=\left(\frac{a}{c}\right)w$; $\frac{n}{w}=\frac{b}{c}$, or $n=\left(\frac{b}{c}\right)w$; $\frac{p}{n}=\frac{a}{b}$.

In other words, if $a{:}b$, for example, equals 1:2, then the pull down the plane is half as great as the pressure exerted perpendicularly against the plane; and so on.

Similar Rectangles. Triangles, of course, are not the only geometric figures which may be similar in shape. Thus if the corresponding sides of two rectangles are in proportion, the rectangles are said to be similar. Here



$$\frac{w}{l}=\frac{w'}{l'}, \text{ or } l\frac{w}{w'}=\frac{l}{l'}.$$

Moreover, the perimeters of two similar rectangles are in the same ratio as any two corresponding sides; i.e.,

$$\frac{P}{P'}=\frac{l}{l'}=\frac{w}{w'}, \text{ or}$$

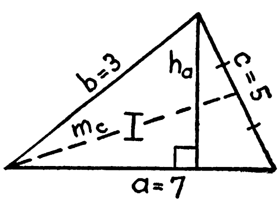
$$\tfrac{32}{24}=\tfrac{12}{9}=\tfrac{4}{3}.$$

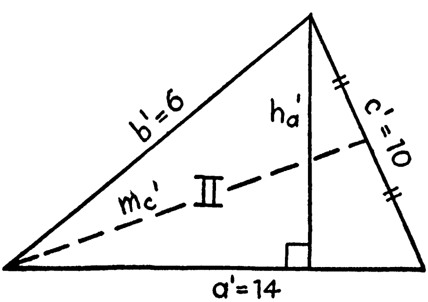
It should also be noted that any two squares are "automatically" similar, and that therefore their perimeters are also in the same ratio as the corresponding sides of the squares.

General Properties of Similar Triangles. This property of the perimeters of similar rectangles holds true of similar triangles also; in fact, in two similar triangles, *any two corresponding "parts"*—altitudes, medians, angle-bisectors—have the same ratio as a pair of corresponding sides; thus in similar triangles I and II we have:

$$\frac{P}{P'}=\frac{ha}{ha'}=\frac{mc}{mc'}=\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\tfrac{1}{2};$$

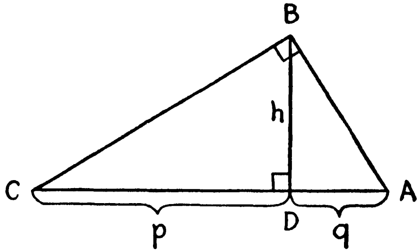
the same ratio (1:2) would likewise hold for any other pair of corresponding altitudes or corresponding medians.



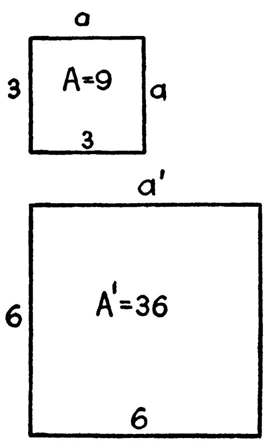


Exercise 68.

1. Two similar rectangles have respective bases of 6″ and 15″. If the altitude of the first is 10″, what is the perimeter of the second rectangle?
2. The base of a triangle is 6 ft. and its altitude is 4 ft. If the corresponding base of a similar triangle is 18 ft., what is the area of the second triangle?



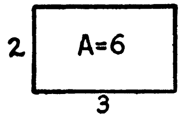
3. If AD=8 and AC=18, find the length of AB; the length of BD.
4. If h=12 and q=9, find CA and CB.
5. A tree casts a shadow 90 ft. long at the same time that a nearby post 6 ft. high casts a shadow 4 ft. long. How high is the tree?
6. The base of a triangle is 20″, and the other sides are 10″ and 16″. A line parallel to the base cuts off 2″ from the lower end of the shorter side. Find (a) the segments of the other side, and (b) the length of the parallel line.
7. The sides of a right triangle are 9″, 12″ and 15″. Find (a) the altitude to the hypotenuse, and (b) the segments of the hypotenuse into which this altitude divides it.
8. The bases of a trapezoid are 6″ and 8″, and the other two sides are 4″ and 5″ respectively. How far must each of these latter sides be produced before they meet in a point?



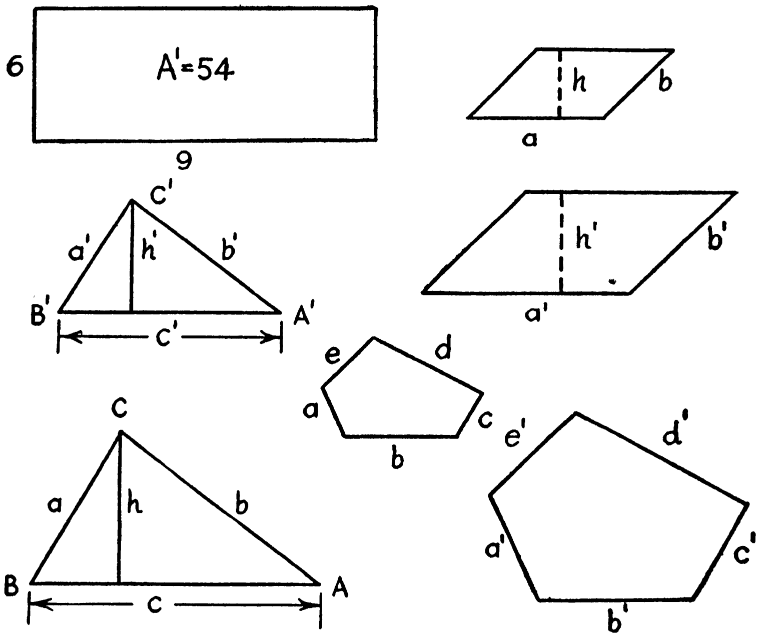
Areas of Similar Figures. It will be noted that in the two squares here shown, the ratio of the sides is 1:2, while the ratio of their *areas* is 1:4; or, doubling the side multiplies the area by *four,* and *not* by two. Similarly, tripling the sides of a rectangle multiplies the area by 9 instead of by 3. In short, the *areas of any two similar figures* are to each other as the *squares* of their respective sides, or the squares of any of their corresponding dimensions. Thus,

$$\frac{\text{Area}\triangle\ \text{ABC}}{\text{Area}\triangle\text{A'B'C'}}=\frac{a^2}{a'^2}=\frac{b^2}{b'^2}=\frac{c^2}{c'^2}=\frac{h^2}{h'^2}=\frac{P^2}{P'^2},$$

where P and P' represent their perimeters. So also in the case of all similar polygons, i.e., polygons having their respective angles

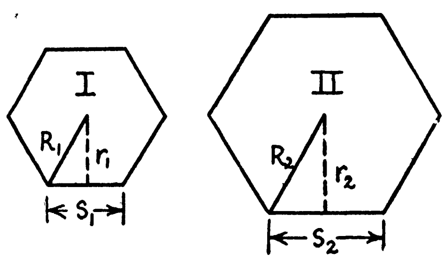


equal and their corresponding sides proportional: the areas of such polygons are to each other as the squares of their corresponding sides, altitudes, etc.



Regular Polygons; Circles. It is easily seen that if any two *regular* polygons have the same number of sides they must be similar. The *perimeters* of any two such regular polygons are to each other as any two corresponding sides; so also is the ratio of their corresponding radii and apothems. Expressed symbolically, we have:

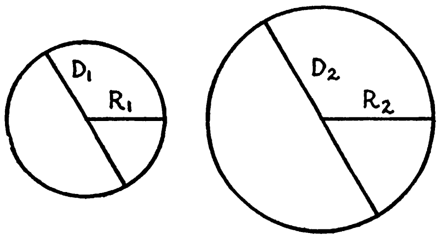
$\frac{P_I}{P_{II}} = \frac{s_1}{s_2} = \frac{R_1}{R_2} = \frac{r_1}{r_2}$, where P_I and P_{II} denote their perimeters, R_1 and R_2 their radii, r_1 and r_2 their apothems, and s_1 and s_2 their sides, respectively.



The *areas* of two regular polygons having the same number of sides are in the same ratio as the *squares* of their corresponding parts: thus

$$\frac{A_I}{A_{II}}=\frac{s_1^2}{s_2^2}=\frac{R_1^2}{R_2^2}=\frac{r_1^2}{r_2^2}=\frac{P_I^2}{P_{II}^2}.$$

In the case of circles the relations are very much the same; in a sense, any two circles may be regarded as "similar figures." The perimeters (i.e., circumferences) of any two circles are to each other as their respective radii or diameters; their areas, however, are to each other as the squares of their corresponding radii, diameters, or circumferences; or, $\frac{C_1}{C_2}=\frac{R_1}{R_2}=\frac{D_1}{D_2}$, and $\frac{A_1}{A_2}=\frac{R_1^2}{R_2^2}=\frac{D_1^2}{D_2^2}=\frac{C_1^2}{C_2^2}$.



EXAMPLE 1: The distances across the flats of two hexagonal plates are 1″ and 1½″, respectively. What is the ratio of their cross-sectional areas?

SOLUTION: $\frac{A_1}{A_2}=\frac{(1)^2}{(1\frac{1}{2})^2}=\frac{\frac{1}{9}}{4}=\frac{4}{9}$, *Ans.*

EXAMPLE 2: Two circular discs are 3″ and 5″ in diameter, respectively. How many times larger than the first disc is the second?

SOLUTION: $\frac{A_2}{A_1}=\frac{5^2}{3^2}=\frac{25}{9}=2\frac{7}{9}$ times as large, *Ans.*

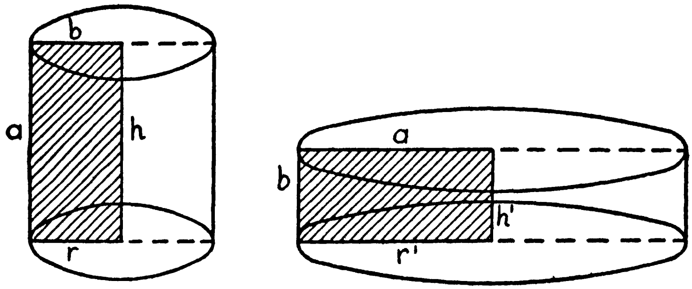
EXAMPLE 3: The area of one circular plate is twice that of another. Find the ratio of their circumferences.

SOLUTION: $\frac{A_1}{A_2}=\frac{2}{1}=\frac{R_1^2}{R_2^2}=\left(\frac{R_1}{R_2}\right)^2$

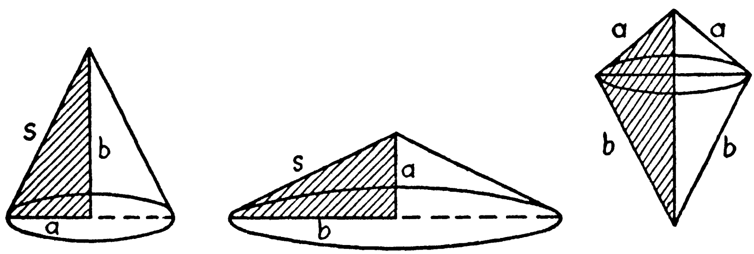
$$\left(\frac{R_1}{R_2}\right)=\sqrt{\frac{2}{1}}=\sqrt{2}=1.41$$

$$\frac{C_1}{C_2}=\frac{R_1}{R_2}=1.41, \textit{ Ans.}$$

Similar Solids of Revolution. A right circular cylinder may be regarded as having been formed by revolving a rectangle about either of its sides as an axis of rotation; the side of the rectangle about which it is revolved becomes the altitude of the cylinder, and the other side becomes the radius of the base. In the same manner, a right circular cone can be regarded as the solid that is formed by revolving a right triangle about

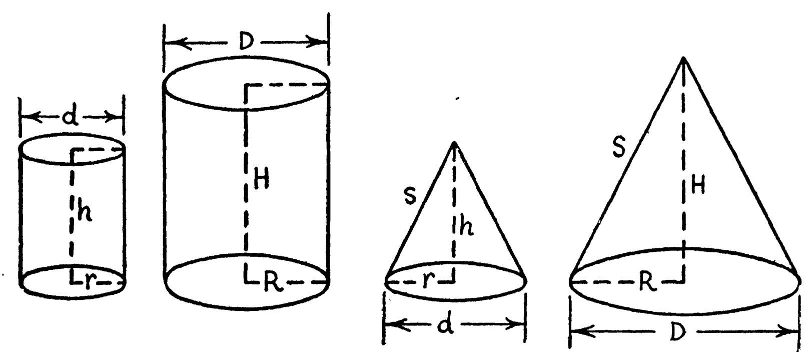


either of its sides as an axis; the side around which it is revolved becomes the altitude, the other side becomes the radius of the base, and the hypotenuse becomes the slant height. If a right triangle is revolved around



the hypotenuse as an axis, two circular cones are formed (one inverted), having a common base, and a *combined altitude* equal to the hypotenuse.

If now we consider cylinders and cones of revolution formed from similar rectangles and similar right triangles, respectively, we have *similar solids of revolution;* their *areas,* lateral and total, are to each other as the *squares* of their corresponding dimensions, and their *volumes* are to each other as the *cubes* of their corresponding dimensions.



I II I II

Thus for similar cylinders of revolution:

$$\frac{\text{L.A.}_{\text{I}}}{\text{L.A.}_{\text{II}}}=\frac{\text{T.A.}_{\text{I}}}{\text{T.A.}_{\text{II}}}=\frac{h^2}{H^2}=\frac{r^2}{R^2}=\frac{d^2}{D^2};$$

$$\frac{\text{V}_{\text{I}}}{\text{V}_{\text{II}}}=\frac{h^3}{H^3}=\frac{r^3}{R^3}=\frac{d^3}{D^3}.$$

And for similar cones of revolution:

$$\frac{\text{L.A.}_{\text{I}}}{\text{L.A.}_{\text{II}}}=\frac{\text{T.A.}_{\text{I}}}{\text{T.A.}_{\text{II}}}=\frac{h^2}{H^2}=\frac{r^2}{R^2}=\frac{d^2}{D^2}=\frac{s^2}{S^2};$$

$$\frac{\text{V}_{\text{I}}}{\text{V}_{\text{II}}}=\frac{h^3}{H^3}=\frac{r^3}{R^3}=\frac{d^3}{D^3}=\frac{s^3}{S^3}.$$

Spheres. Just as any two circles may be regarded as similar, so any two spheres may be regarded as similar. Therefore we have

$$\frac{\text{A}_{\text{I}}}{\text{A}_{\text{II}}}=\frac{r^2}{R^2}=\frac{d^2}{D^2},\ \text{and}\ \frac{\text{V}_{\text{I}}}{\text{V}_{\text{II}}}=\frac{r^3}{R^3}=\frac{d^3}{D^3}.$$

Example 1: Two cylindrical tin cans are similar, i.e., their heights and diameters are in the proportion of 2:3. Find (a) the ratio of their lateral surfaces; (b) the ratio of their volumes.

Solution: $\frac{\text{L.A.}_{\text{I}}}{\text{L.A.}_{\text{II}}}=\frac{2^2}{3^2}=4/9$, *Ans.*

$\frac{\text{V}_{\text{I}}}{\text{V}_{\text{II}}}=\frac{2^3}{3^3}=8/27$, *Ans.*

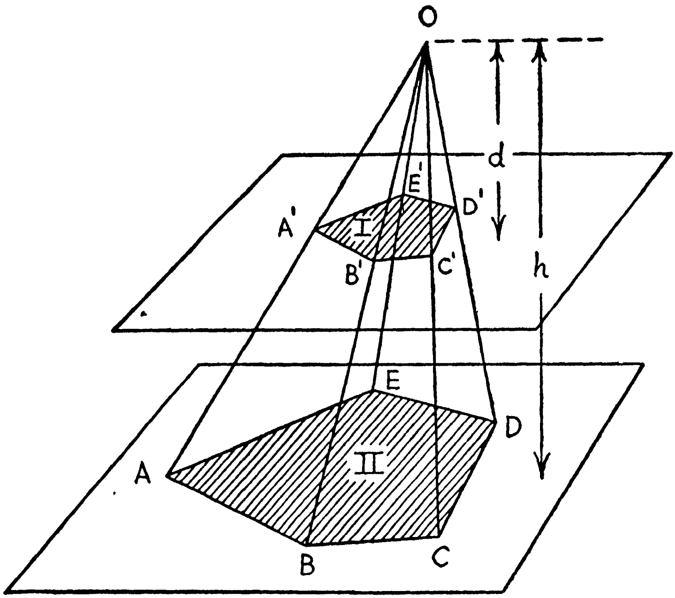
Example 2: A right circular cone has a diameter of 4″ and an altitude of 5″. Another cone, similar to the first, has an altitude of 10″. Find (a) the diameter of the second cone; (b) the ratio of their total areas; (c) how many times larger in volume the second cone is than the first.

Solution: (a) $4/5=\frac{x}{10}$; $5x=40$; $x=8$, *Ans.*

(b) $\frac{\text{T.A.}_{\text{I}}}{\text{T.A.}_{\text{II}}}=\frac{4^2}{5^2}=16/25$, *Ans.*

(c) $\frac{\text{V}_{\text{II}}}{\text{V}_{\text{I}}}=\frac{5^3}{4^3}=125/64=1\,61/64$ times as large, *Ans.*

Section of Pyramid Parallel to the Base. If in any pyramid, whether a regular pyramid or an oblique pyramid, a plane section is passed parallel to the base, the polygon formed is similar to the base, and the pyramid



"cut off" is similar to the original pyramid. Furthermore, the lateral edges will be divided in the ratio of $d:(h-d)$, where d is the distance of the cutting plane below the vertex, and h is the original altitude. Or:

$$\frac{OA'}{OA}=\frac{OB'}{OB}=\frac{OC'}{OC}(\text{etc.})=\frac{d}{h};$$

also, $$\frac{A'B'}{AB}=\frac{B'C'}{BC}=\frac{C'D'}{CD}=(\text{etc.})=\frac{d}{h};\text{ and}$$

$$\frac{\text{area I}}{\text{area II}}=\frac{(OA')^2}{(OA)^2}=\frac{(A'B')^2}{(AB)^2}=\frac{d^2}{h^2}.$$

Exercise 69.

1. If the radius of a circle is cut in half, how is its circumference changed? its area?
2. The areas of two circles are as 4:9; if the radius of the larger circle is 12 inches, what is the circumference of the smaller circle?
3. If the edge of a cube is doubled, how is its area changed? its volume?
4. The diameters of two circles are 6″ and 8″ respectively. Find (a) the ratio of their circumference, and (b) the ratio of their areas.
5. The circumference of a circle is 20″. What is the circumference of a circle having twice the area of the given circle?

6. If the diameter of a cylindrical shaft is diminished by 10%, how is its circumference changed? the area of its cross section?
7. A water storage tank is fed by two pipe lines, one 8″ and one 6″ in diameter. If these are to be replaced by a single pipe line having the same capacity as the two combined, what diameter pipe should be used? (*Hint:* $\pi R^2 + \pi r^2 = \pi x^2$.)
8. A rectangular zinc cut for photoengraving is reduced "three-fourths"; what is the ratio of the areas involved?
9. If the diameter of a water main is made half again as large, what is the per cent of increase in the capacity of the pipe? (*Hint:* compare the cross-sectional areas.)
10. Find the ratio of the volume of two spheres if their areas are in the ratio of 3:1.
11. Find the lateral area and the volume of a cone of revolution if the radius of the base is 8″ and the slant height forms an angle of 60° with the plane of the base.
12. A cylindrical container 8″ high has a diameter of 4″. If each dimension is increased by 25%, what is the ratio of the new total surface to the original total surface? What is the per cent of increase in total surface?
13. An equilateral triangle whose side is 6″ is revolved about one of its sides as an axis. Find the total surface and volume of the solid generated.
14. The area of the base of a circular cone is 108 sq. in., and its altitude is 6″. If a section is passed parallel to the base and 4″ above it, what is the area of the base of the cone cut off?
15. If a cylindrical metal drum used for shipping chemicals has each dimension increased by 20%, what is the per cent of increase in its capacity?
16. If the inside diameter of a pipe is increased 50%, how much more water will flow through it at the same rate in the same amount of time?
17. If the height of a cylinder is cut in half, and the diameter is doubled, what is the change in volume?
18. A micro-photograph was enlarged "20 diameters." How many times larger than the original is the area of the photograph?
19. One of two circular water pipes is 4″ in diameter, and the other is 8″. (a) How many times larger is the cross-sectional area of the second? (b) How much more water will flow through the second pipe?
20. Because of cloudy weather a photographer enlarged the diaphragm (circular opening) of his camera from a diameter of 0.4 cm. to 0.6 cm. By what per cent did this increase the area of the opening?

TABLE OF TANGENTS, COSINES, AND SINES

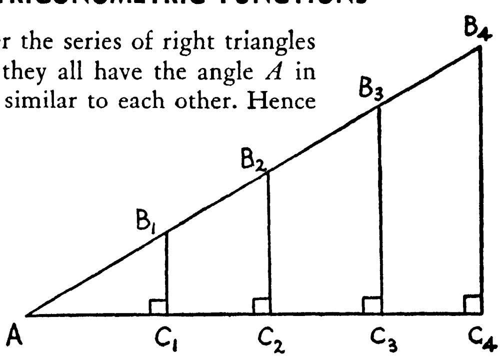
Angle	Tangent (opp./adj.)	Cosine (adj./hyp.)	Sine (opp./hyp.)	Angle	Tangent (opp./adj.)	Cosine (adj./hyp.)	Sine (opp./hyp.)
0°	.0000	1.0000	.0000	45°	1.0000	.7071	.7071
1°	.0175	.9998	.0175	46°	1.0355	.6947	.7193
2°	.0349	.9994	.0349	47°	1.0724	.6820	.7314
3°	.0524	.9986	.0523	48°	1.1106	.6691	.7431
4°	.0699	.9976	.0698	49°	1.1504	.6561	.7547
5°	.0875	.9962	.0872	50°	1.1918	.6428	.7660
6°	.1051	.9945	.1045	51°	1.2349	.6293	.7771
7°	.1228	.9925	.1219	52°	1.2799	.6157	.7880
8°	.1405	.9903	.1392	53°	1.3270	.6018	.7986
9°	.1584	.9877	.1564	54°	1.3764	.5878	.8090
10°	.1763	.9848	.1736	55°	1.4281	.5736	.8192
11°	.1944	.9816	.1908	56°	1.4826	.5592	.8290
12°	.2126	.9781	.2079	57°	1.5399	.5446	.8387
13°	.2309	.9744	.2250	58°	1.6003	.5299	.8480
14°	.2493	.9703	.2419	59°	1.6643	.5150	.8572
15°	.2679	.9659	.2588	60°	1.7321	.5000	.8660
16°	.2867	.9613	.2756	61°	1.8040	.4848	.8746
17°	.3057	.9563	.2924	62°	1.8807	.4695	.8829
18°	.3249	.9511	.3090	63°	1.9626	.4540	.8910
19°	.3443	.9455	.3256	64°	2.0503	.4384	.8988
20°	.3640	.9397	.3420	65°	2.1445	.4226	.9063
21°	.3839	.9336	.3584	66°	2.2460	.4067	.9135
22°	.4040	.9272	.3746	67°	2.3559	.3907	.9205
23°	.4245	.9205	.3907	68°	2.4751	.3746	.9272
24°	.4452	.9135	.4067	69°	2.6051	.3584	.9336
25°	.4663	.9063	.4226	70°	2.7475	.3420	.9397
26°	.4877	.8988	.4384	71°	2.9042	.3256	.9455
27°	.5095	.8910	.4540	72°	3.0777	.3090	.9511
28°	.5317	.8829	.4695	73°	3.2709	.2924	.9563
29°	.5543	.8746	.4848	74°	3.4874	.2756	.9613
30°	.5774	.8660	.5000	75°	3.7321	.2588	.9659
31°	.6009	.8572	.5150	76°	4.0108	.2419	.9703
32°	.6249	.8480	.5299	77°	4.3315	.2250	.9744
33°	.6494	.8387	.5446	78°	4.7046	.2079	.9781
34°	.6745	.8290	.5592	79°	5.1446	.1908	.9816
35°	.7002	.8192	.5736	80°	5.6713	.1736	.9848
36°	.7265	.8090	.5878	81°	6.3138	.1564	.9877
37°	.7536	.7986	.6018	82°	7.1154	.1392	.9903
38°	.7813	.7880	.6157	83°	8.1443	.1219	.9925
39°	.8098	.7771	.6293	84°	9.5144	.1045	.9945
40°	.8391	.7660	.6428	85°	11.4301	.0872	.9962
41°	.8693	.7547	.6561	86°	14.3007	.0698	.9976
42°	.9004	.7431	.6691	87°	19.0811	.0523	.9986
43°	.9325	.7314	.6820	88°	28.6363	.0349	.9994
44°	.9657	.7193	.6947	89°	57.2900	.0175	.9998
45°	1.0000	.7071	.7071	90°		.0000	1.0000

CHAPTER IV

SHOP TRIGONOMETRY

William L. Schaaf

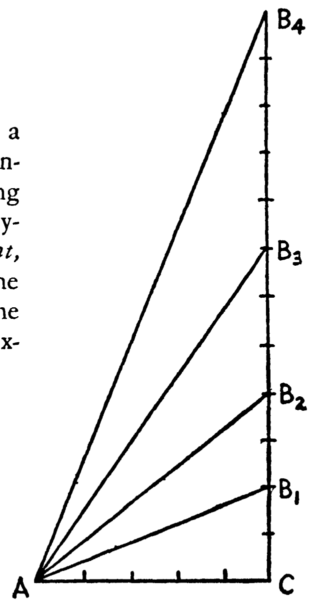
20. USING TRIGONOMETRIC FUNCTIONS



Similar Triangles. Consider the series of right triangles AB_1C_1, AB_2C_2, etc. Since they all have the angle A in common, the triangles are similar to each other. Hence the ratios of the corresponding pairs of sides are equal; or

$$\frac{B_1C_1}{AC_1}=\frac{B_2C_2}{AC_2}=\frac{B_3C_3}{AC_3}=\frac{B_4C_4}{AC_4}=k,$$

where k represents some numerically constant value, independent of the units of measure used. In other words, the value of k could be used as a *measure of angle A*. Furthermore, if we consider another series of right triangles having one side fixed, but with the angle at A varying, then the series of ratios is *not constant*, but varies with the size of the angle *A;* if the angle increases, the ratio increases, and if the angle decreases, the ratio decreases. For example:



$$\frac{B_1C}{AC}=2/5=.4;\quad \frac{B_3C}{AC}=7/5=1.4;$$

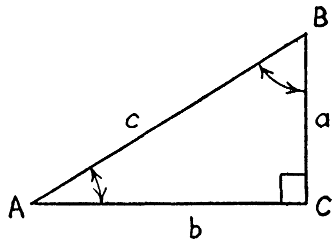
$$\frac{B_2C}{AC}=4/5=.8;\quad \frac{B_4C}{AC}=12/5=2.4;\ \text{etc.}$$

In any right triangle, the ratio of the side opposite an acute angle to the side adjacent to it is called *the tangent of that angle;* this is written:

$$\tan A = \frac{BC}{AC} = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}.$$

Similarly:

$$\tan B = \frac{AC}{BC} = \frac{b}{a}.$$



Trigonometric Ratios. In the same way, any other pair of sides of the right triangle might be used to form a ratio which can be regarded as a measure of the angle. Such ratios are called *trigonometric functions;* their names and definitions are given below:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{b}$$

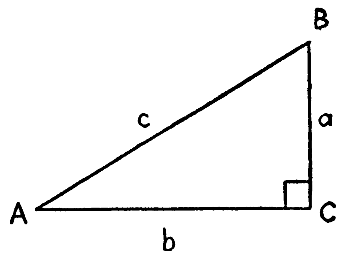
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c} \qquad \csc A = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{c}{a}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b} \qquad \cot A = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{b}{a}$$

It will be sufficient for most practical purposes to use the three functions in the first column, viz., *sine, cosine* and *tangent;* these should by all means be memorized. The other three, the *secant, cosecant,* and *cotangent,* are perhaps not so important for ordinary use.

Table of Natural Functions. The numerical values of these ratios for various angles from 1° to 90° have been carefully worked out to several decimal places; part of such a table is given above for the three most commonly used functions—the sine, the cosine, and the tangent. How such a table of the values of the various functions is used will be shown as we go along.

Co-functions. By studying the diagram and referring to the definitions once more, the following relations will be seen to hold:



$$\sin A = \frac{a}{c} = \cos B,$$

$$\text{and } \cos A = \frac{b}{c} = \sin B.$$

In other words, since $\angle A + \angle B = 90°$, the sine of any angle equals the cosine of its complement, and the cosine of an angle equals the sine of its complement. This explains the meaning of the word cosine, i.e., *co-sine,* or "complement's sine." Thus if sin 32° = .5299, then cos 58° also equals .5299; etc. Similarly,

$$\tan A = \frac{a}{b} = \cot B;$$

$$\cot A = \frac{b}{a} = \tan B;$$

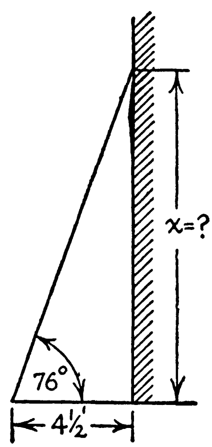
and also:

$$\sec A = \frac{c}{b} = \csc B;$$

$$\csc A = \frac{c}{a} = \sec B.$$

Using the Tangent. Several common types of problems arise where the use of the tangent is very convenient.

EXAMPLE 1: A ladder leaning against the wall of a building makes an angle of 76° with the ground. If the foot of the ladder is 4½ ft. from the base of the wall, how high above the ground is the point on the wall where the top of the ladder touches it?



SOLUTION:

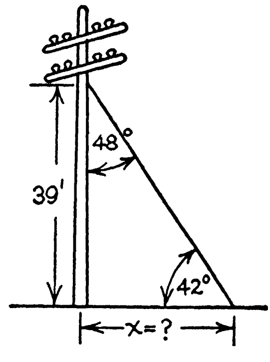
$$\tan 76° = \frac{x}{4.5} \text{ (from the figure)}$$

$$\tan 76° = 4.0108 \text{ (from the table)}$$

$$\text{hence } \frac{x}{4.5} = 4.0108$$

$$\text{or } x = (4.5)(4.0108) = 18.05 \text{ ft., } \textit{Ans.}$$

EXAMPLE 2: The guy wire supporting a telegraph pole is fastened to the pole at a point 39 ft. above the ground, and makes an angle of 42° with the ground. Find how far from the base of the pole must the stake be driven to fasten the guy wire.

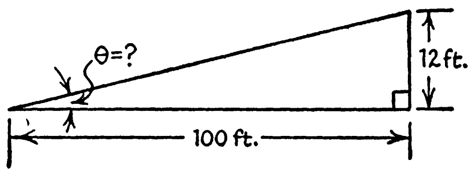


SOLUTION:

$$\frac{x}{39}=\tan 48° = 1.1106$$

$$x=(39)(1.1106)=43.3 \text{ ft., } \textit{Ans.}$$

EXAMPLE 3: A ramp rises 12 ft. in a horizontal distance of 100 ft. Find the angle (θ) of inclination.



SOLUTION:

$\tan \theta = {}^{12}\!/_{100} = .1200$

From the table: $\tan 7° = .1228$

$\theta = 7°$ (approx.), *Ans.*

NOTE: More exact results can be found by interpolation similar to that used in finding logarithms, or by using a more complete table of trigonometric functions showing values for degrees and minutes. Such a table is given at the back of this book. Thus, by using the complete table, the angle in Ex. 3, expressed to the nearest minute, is 6° 51′.

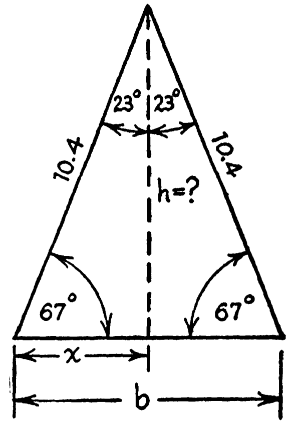
Exercise 70.

In the following problems, use the table of values at the back of the book when finding angles, obtaining your result to the nearest minute.

1. Find the side of an equilateral triangle whose altitude is 22.8″.
2. The slope of a roof is 4 inches in each horizontal foot. What angle does it make with the horizontal?
3. The altitude of an isosceles triangle is 12 and the base is 4. Find the length of the equal sides and the angles of the triangle.
4. A railroad track makes an angle of 10° with the horizontal. How many feet does it rise in 1000 ft. along the horizontal?
5. Find the apothem of a regular octagon whose sides are 4″ each in length.
6. A rectangle is 36″×48″. Find the angle between the diagonal and the longer side.

Using Sines and Cosines. How these functions are similarly utilized in practical problems involving right triangles will now be illustrated.

EXAMPLE 1: Find the altitude of an isosceles triangle whose vertex angle is 46° and whose equal sides are each 10.4 inches. What is the length of its base?



SOLUTION: $\frac{h}{10.4}=\sin 67°=.9205$

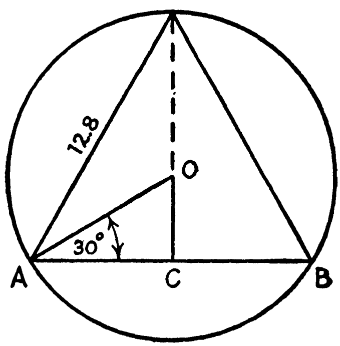
$h=(10.4)(.9205)=9.57$, *Ans.*

$\frac{x}{10.4}=\cos 67°=.3907$

$x=(10.4)(.3907)=4.06$

$b=(2)(4.06)=8.12$, *Ans.*

EXAMPLE 2: Find the radius of the circle circumscribed around an equilateral triangle 12.8″ on a side.



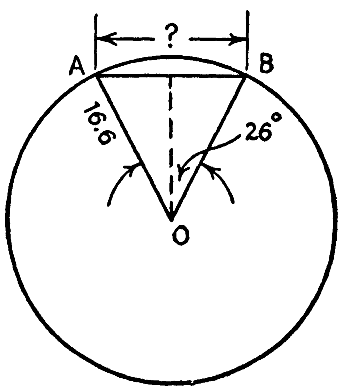
SOLUTION:

$AC=\frac{1}{2}(12.8)=6.4$

$\frac{AC}{r}=\cos 30°$

$r=\frac{AC}{\cos 30°}=\frac{6.4}{.8660}=7.39$, *Ans.*

EXAMPLE 3: How long is a chord of a circle that subtends an angle of 26° at the center, if the radius of the circle is 16.6 inches? What is the distance of the chord from the center?



SOLUTION: $\frac{AM}{16.6}=\sin 13°=.2250$

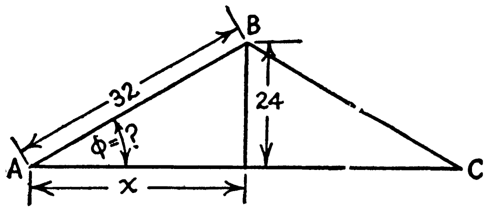
$AM=(16.6)(.2250)=3.74$

$\text{chord}=2\times(AM)=7.48$, *Ans.*

$\frac{OM}{16.6}=\cos 13°=.9744$

$OM=(16.6)(.9744)=16.18$, *Ans.*

Example 4: The equal sides of an isosceles triangle are each 32″ and the altitude is 24″. Find the base and the angles.



Solution:

$$\sin \phi = \frac{24}{32} = .7500$$

$$\phi = 48°35', \textit{ Ans.}$$

$$\frac{x}{32} = \cos 48°35' = .6615$$

$$x = (32)(.6615) = 21.17$$

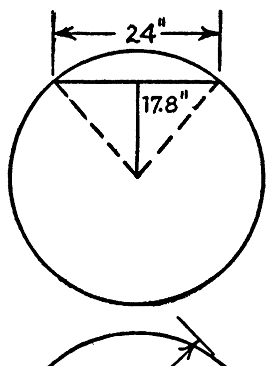
$$AC = 2x = (2)(21.17) = 42.34, \textit{ Ans.}$$

Exercise 71.

In the following problems, use the table in the back of the book when finding angles, obtaining the answer to the nearest minute.

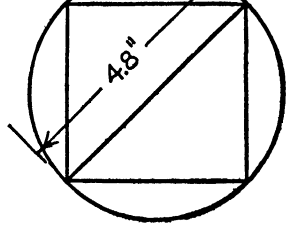
1. Find the radius of a circle if a 12-inch chord subtends an angle of 28° at the center.

2. The equal sides of an isosceles triangle are 22″ long and the altitude is 9.2″. Find the base and the angles.



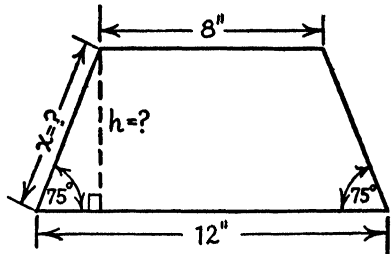
3. A chord 24 inches long is 17.8 inches distant from the center. Find (a) the radius of the circle, and (b) the subtended angle of the chord.

4. What is the largest square that can be milled from a circular disc 4.8 inches in diameter?



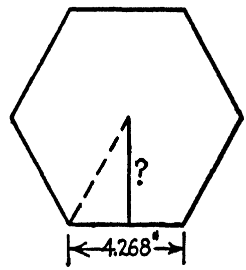
5. An isosceles triangle has sides of 16″, 16″ and 10″. Find the angles and the altitude.

6. The bases of an isosceles trapezoid are 8″ and 12″. If the base angles are 75°, find the equal sides and the altitude.

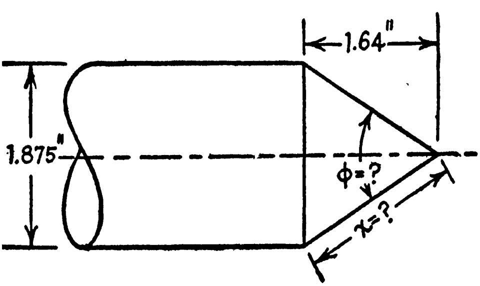


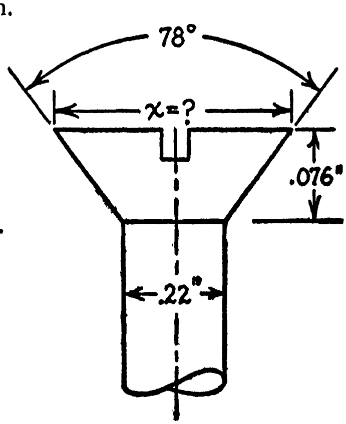
7. If a chord is 18½″ long and the radius of the circle is 14.2″, find the distance of the chord from the center and the central angle.

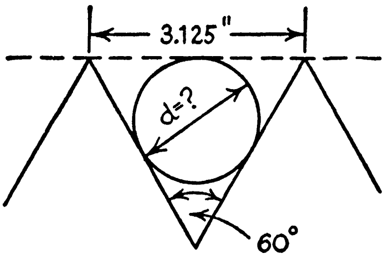
8. Find the perpendicular distance from the center to the sides of a regular hexagon whose sides are each 4.268″ long?

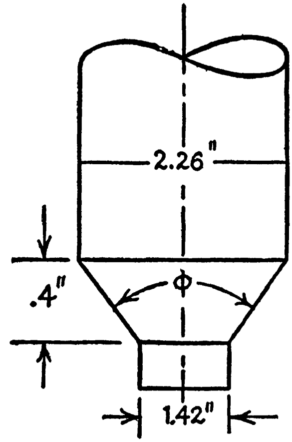


9. Find the radius of a circle inscribed in an equilateral triangle whose perimeter is 45 inches.
10. The side of a regular pentagon is 18.6 inches. Find the radius of the inscribed and circumscribed circles, respectively.
11. Gable rafters 19 ft. long project 1½ ft. beyond the walls of a house and are set with a pitch (angle with the horizontal) of 38°. Find the height of the ridgepole and the width of the house.
12. Find the value of ϕ and x from the dimensions given.
13. Find x in the flat-head screw as shown.





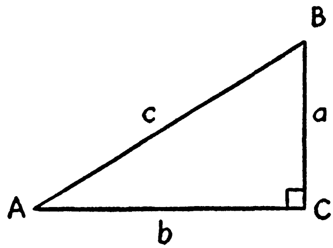




14. Find the diameter d of the wire laid in the groove of the screw thread as shown.
15. Find the angle ϕ in the spindle as shown.

21. PRACTICAL APPLICATIONS OF RIGHT TRIANGLES

Solution of Right Triangles. From the foregoing discussion it should be clear that a right triangle is completely determined if a side and any other part are known. In other words, given the length of a side and any other part, all the other parts can then be found; this is called solving the right triangle. By way of summary, the solution of a right triangle may be effected by the use of one or more of the following fundamental relationships:



(1) $a^2+b^2=c^2$

(2) $\sin A=\dfrac{a}{c}=\cos B$

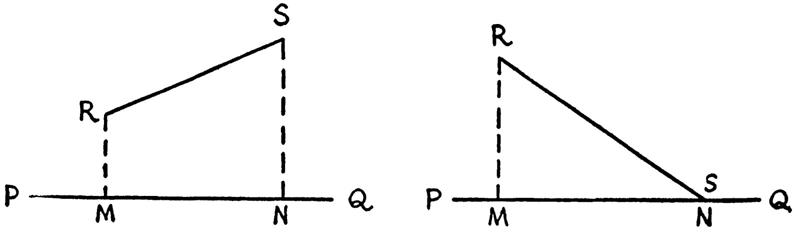
(3) $\cos A=\dfrac{b}{c}=\sin B$

(4) $\tan A=\dfrac{a}{b}=\cot B$

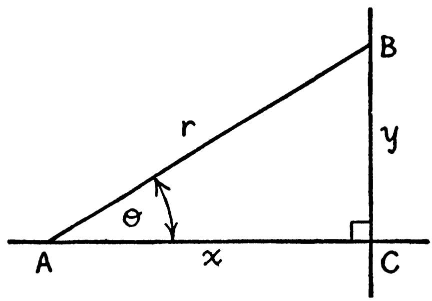
(5) $A+B=90°$

The rest of the present section will deal with practical uses of right triangles. Standard notation, i.e., side "a" opposite $\angle A$, side "b" opposite $\angle B$, etc., $\angle C$=right angle, will be used consistently throughout.

Projections. In physics and mechanics frequent use is made of horizontal and vertical projections. As we saw in Chapter III, section 13, the projection of one line segment upon another was the segment on the



second line between the feet of the perpendiculars drawn to the second line from the ends of the first line. Thus, the projection of RS upon PQ in each case here shown is the segment MN. If now, we project an oblique line, such as AB or r, upon each of two mutually perpendicular axes, respectively, then the projection of AB upon the horizontal axis is AC, or x; and the projection of AB upon the vertical axis is BC, or y. These projections, AC and BC, can be expressed as follows:



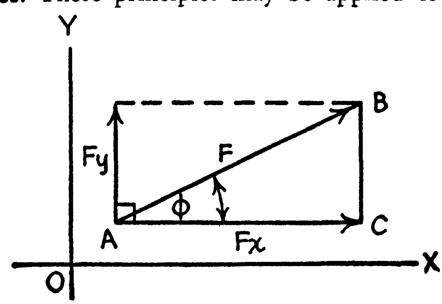
$$AC = x = r \cdot \cos \theta,$$
$$\text{and } BC = y = r \cdot \sin \theta.$$

This leads to two simple but important rules concerning projections:

Rule 1: The horizontal projection of any line segment equals the length of the segment multiplied by the cosine of the angle of inclination, or the angle with the horizontal.

Rule 2: The vertical projection of any line segment equals the length of the segment multiplied by the sine of the angle of inclination.

Component Forces and Velocities. These principles may be applied to the component parts of forces or velocities acting obliquely to the horizontal and vertical directions. Thus if AB represents a force F acting at an angle ϕ to the horizontal, the two perpendicular component forces F_x and F_y are equivalent, when considered jointly, to the single original force F.

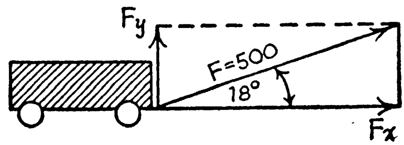


Furthermore, from the preceding paragraph, we now see that

$$F_x = F \cos \phi,$$
$$\text{and } F_y = F \sin \phi.$$

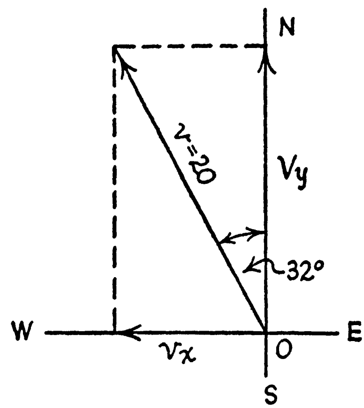
It is also clear that $F_x^2 + F_y^2 = F^2$. From these relations, many simple problems concerning forces and velocities can be solved, as will be shown below.

EXAMPLE 1: A pull of 500 lb. is applied to a cart at an angle of 18° to the horizontal. What is the effective horizontal pull? How much is the force that tends to lift the cart vertically upward?



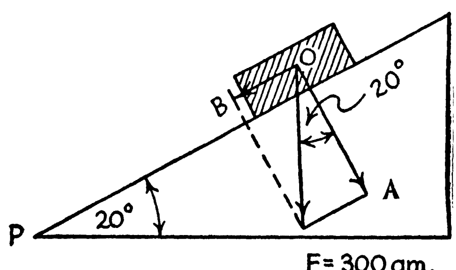
SOLUTION: $F_x = 500(\cos 18°) = 500(.9511) = 476.6$ lb., *Ans.*
$F_y = 500(\sin 18°) = 500(.3090) = 154.5$ lb., *Ans.*

EXAMPLE 2: A ship steering a course 32° west of north is moving at 20 miles an hour. Where will it be one hour after it leaves point O?



SOLUTION: $v_x = 20(\cos 58°) = 20(.5299) = 10.6$ mi. west of point O, *Ans.*
$v_y = 20(\sin 58°) = 20(.8480) = 17.0$ mi. north of point O, *Ans.*

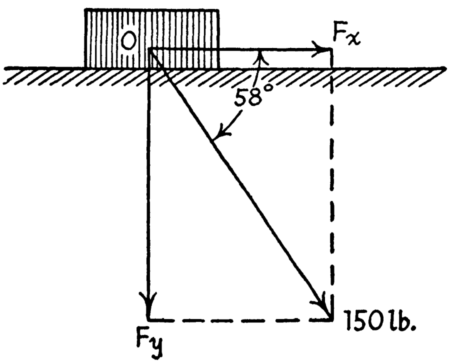
EXAMPLE 3: A block rests upon an incline of 20°. If the block weighs 300 gm., find (a) the pressure exerted perpendicularly upon the inclined plane; and (b) the pull parallel to the plane, tending to cause it to slide down the plane.



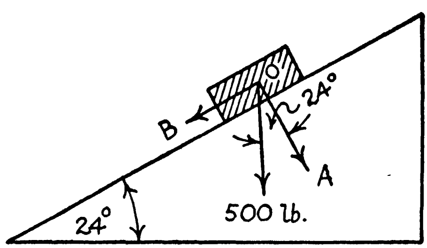
SOLUTION: If the angle at P=20°, then ∠AOF also equals 20°. Hence,
OA=300(cos 20°)=300(.9397)=281.9 gm., *Ans.*
OB=300(sin 20°)=300(.3420)=102.6 gm., *Ans.*

Exercise 72.

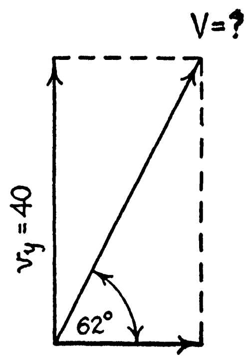
1. An object is moving with a velocity of 200 ft. per minute along a line making an angle of 33° with the horizontal. Find the horizontal component of this velocity.
2. An airplane is flying northeast at the rate of 300 miles per hour. At what rate is it moving eastward? at what rate northward?



3. A force of 150 lb. is applied to a block resting on the horizontal. If the force makes an angle of 58° with the horizontal, what force tends to draw the block horizontally? What force tends to pull it vertically downward?
4. A force of 600 lb. acting in a direction inclined 50° from the vertical is applied to a heavy object. Find (a) the force which tends to move the block horizontally, and (b) the force which tends to lift it vertically.
5. A block resting on an incline of 24° weighs 500 lb. Find (a) the perpendicular pressure (OA) exerted by the block against the inclined plane, and (b) the pull (OB) along the plane.

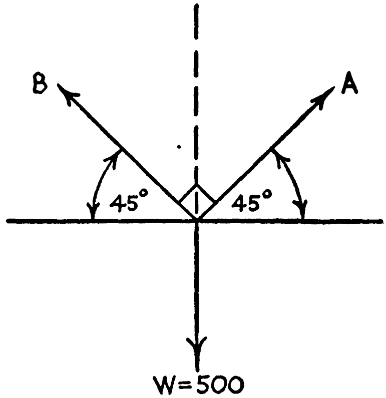


6. Find the velocity of a body moving at an angle of 62° with the horizontal, if the vertical component of its velocity is 40 ft. per second. How fast is it moving horizontally?

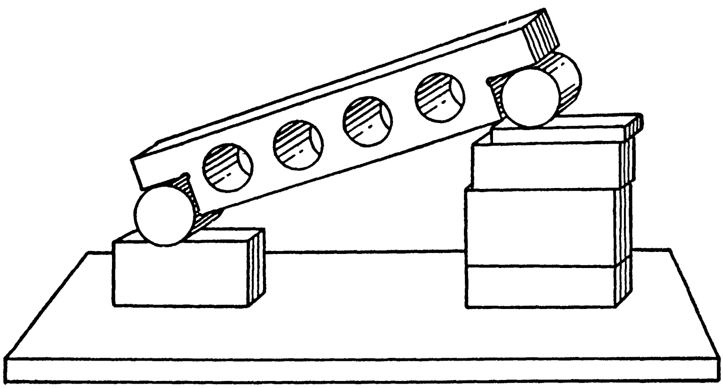


7. The horizontal and vertical components of a force acting on an object are 800 lb. and 600 lb., respectively. Find the original force and its direction of action.

8. A weight *W* of 500 lb. is suspended from point O by two stout chains, OA and OB. If each chain bears half the share of the total weight *W*, find the upward vertical pull exerted by each chain.



The Sine Bar. The sine bar is a device commonly used to facilitate precision angle-measurements. It consists of a very accurately made, heat-treated alloy-steel straight edge, to which are attached two hardened cylinders. All of the surfaces of this straight edge are parallel to each other and to the center line between the two rolls or plugs. These rolls are extremely accurately spaced, and their cylindrical surfaces made square to the measuring surfaces of the bar. The distance between studs on the 5-inch bar is controlled to within ±.0002 inch, and between ±.00025



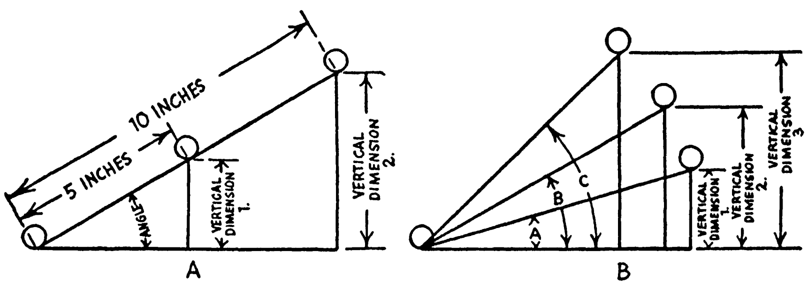
Use of Sine Bar in conjunction with Gage Blocks to obtain precision angular measurement.

inch on the 10-inch bar. These are the two standard sizes of the sine bar. The accompanying illustration shows how the sine bar is used in conjunction with stacks of gage blocks. A 1-inch block is generally used under the lower roll; the stack of blocks under the higher end is adjusted to create the desired angle.

Theory of the Sine Bar. The sine bar makes application of the known relation between the sides of a right triangle and its angles.

The right triangle does not exist physically as a solid triangle, but is partly imaginary. The bar itself is the only part of the triangle that actually exists. The base of the triangle is an imaginary line on the surface plate if the lower button is resting on the plate. The vertical leg of the triangle is an imaginary line from the bottom of the sine bar button to the surface plate. It may take the form of a stack of precision gage blocks if the bar is supported by the blocks, but if the bar is clamped to an angle plate, it will probably be just an imaginary line.

The fundamental relation that holds for all angles in a right triangle is that the ratio of the side opposite the angle to the length of the longest side varies with the size of the triangle.

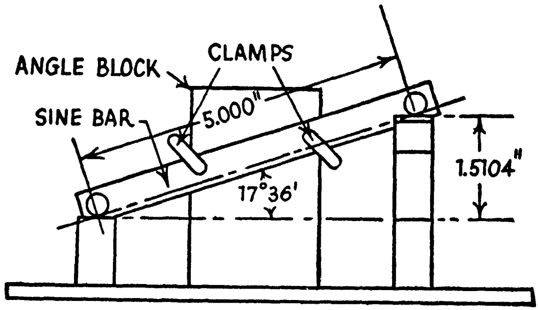


PRINCIPLE OF THE SINE BAR

Now, if the side opposite the right angle is kept constant at some figure, such as 5 inches, then the angle varies directly as the length of the side opposite, and a table of angles and corresponding lengths of opposite sides can be developed. With the aid of such a table, it is necessary to know only the length of the opposite side to determine the angle, and vice versa.

The tables of natural sines found in almost any handbook are developed on the basis of a 1-inch side opposite the right angle, or a 1-inch sine bar. For example, the table of natural sines gives a value of .56184 for the sine of 34° 11′. This means that if the side opposite the right angle is one inch, then the side opposite the angle is .56184 of an inch; and from what has been developed above, it also means that if the side opposite the right angle is 5 inches, then the side opposite the angle is 5 times .56184, or 2.8092 inches. To set up a 5-inch sine bar to an angle of 34° 11′, a stack of gage blocks 2.8092 inches high would be required under one end, with the other resting on the surface plate.

Setting Up an Angle on the Sine Bar. Let it be assumed that a part master has an angle of 17° 36′ to be checked with a sine bar. The sine bar is set up to 17° 36′ by noting the value in any sine bar table and building up a stack of blocks so that the opposite side has this value. If the lower end of the bar rests on a block one inch high, then one inch also must be added to the stack supporting the upper end, because it is the difference between the upper and the lower end which determines the opposite side of the triangle.



SETTING UP A GIVEN ANGLE

Any sine bar table would show that a difference of 1.5104 is necessary to produce an angle of 17° 36′. This means that the high end must be set at 2.5104 off the surface plate. With the bar resting on the blocks, it is clamped to the angle iron.

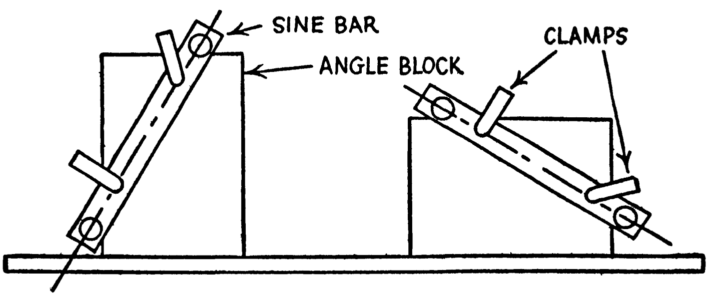
The work is then clamped to the bar so that the 17° 36′ angle on the piece is level with the surface of the plate. The surface is then indicated with a height gage and sensitive lathe indicator to verify the correctness of the angle.

Setting Up to Measure an Unknown Angle. This procedure is similar to setting up for checking a known angle. The angle is measured roughly with a bevel protractor, and the stacks of blocks necessary to produce this angle are made up from the figure read in the table.

The work is clamped to the sine bar after it is set up, and indicated with a height gage and indicator. The error noted is corrected by increasing or decreasing the height of the stack of blocks under the high end. When the indicator shows the surface to be level, the length of the stack is noted. From this, the stack at the lower end is subtracted, and the net value is the opposite side. Referring to a sine bar table, the value nearest this dimension represents the angle measured to the nearest minute. Although it might be possible to compute the number of seconds, for all practical purposes "to the nearest minute" is considered satisfactory.

Applications of the Sine Bar. The application of the sine bar generally is reserved for measurements which require that an angle be determined closer than five minutes, which is the limit of accuracy of the bevel protractor. Of course, with the sine bar it is a simple matter to determine an angle within one minute.

The sine bar is most accurate when measuring small angles, since a small change in the angle at or near the horizontal produces a greater change in the vertical dimension measured by the blocks. As the angle approaches 90 degrees, it is easy to see that the change in height corresponding to a given change in angle is much smaller. For this reason a sine bar should not be set up to measure an angle over 60 degrees, but the setup should be changed so that the complement of the angle can be measured. The complement of the angle is the difference between the angle and 90 degrees, and is obtained by tipping over the angle block to which the sine bar is set up.



MEASURING THE COMPLEMENT

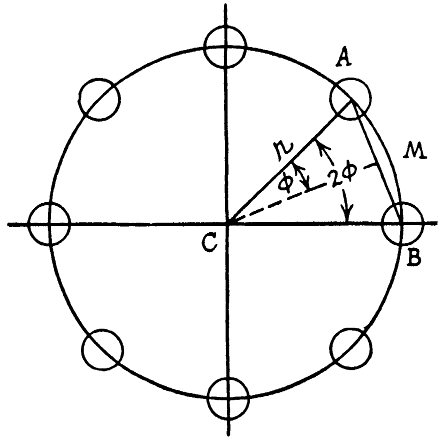
The reason for tipping the bar over and measuring the complement of a large angle instead of the angle directly is illustrated by the following example. An angle of 80 degrees has a sine of .98481, and the opposite side, using a 5-inch bar, would be 4.92405. An angle of 80 degrees, 1 minute would have an opposite side of 4.92430. This means that one minute is represented by a difference of .00025 of an inch in the height of the blocks.

If this angle is turned over on its side it becomes an angle of 10 degrees. A table of sine bar values quickly shows .86825 as the height of the stack of blocks necessary to produce an angle of 10 degrees. An angle of 10 degrees, 1 minute is produced by a stack .86965 of an inch high. One minute is represented by a difference of .0014, which means that at 10 degrees, eliminating the error in the tools themselves, a measurement is more than five times as accurate as one made at 80 degrees. This is the reason why sine bar tables usually stop at 60 degrees, and the inspector or toolmaker using a sine bar will always try to set up an angle over 60 degrees so that he checks the complement instead of the angle itself.

Spacing Holes on a Circle. In order to find the distance from center to center between holes spaced on a circle, it is only necessary to find the length of the chord (AB) joining the centers A and B. If the angle of arc AB is 2ϕ, then clearly,

$$\sin \phi = \frac{AM}{r},$$

or $AM = r \sin \phi$; since $AB = 2(AM)$, then $AB = 2r \sin \phi$; or, in terms of the diameter, the distance between adjacent holes equals $d \sin \phi$. In practice, bolt holes are usually spaced at equal distances from one another; i.e., so that their arcs have equal central angles. The circle drawn through the centers of the holes is called the *bolt circle,* and its diameter is, of course, less than that of the rim or outer edge of the disc, wheel, etc.



EXAMPLE: Find the distance between 12 equally spaced holes on a 20″ circle.

SOLUTION: $360° \div 12 = 30°$,

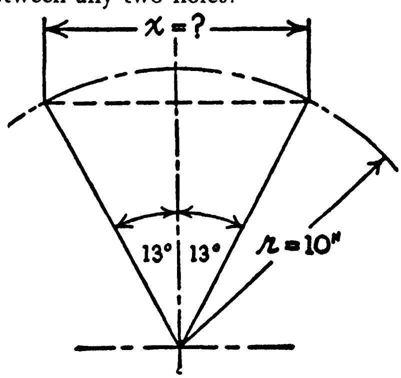
$\frac{1}{2} \times 30° = 15°$.

Hence distance between centers $= d \sin 15°$

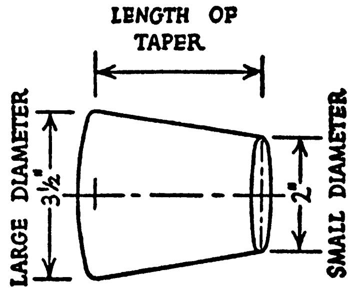
$= 20(.2588) = 5.176''$, *Ans.*

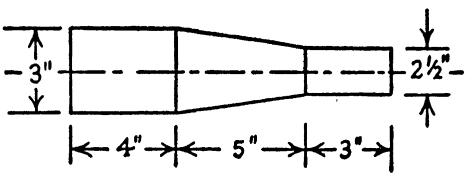
Exercise 73

1. If 8 equally spaced holes are to be drilled on a 14-in. circle, find the distance between the centers of any two adjacent holes.
2. Three holes are to be drilled 120° apart on a 12-in. bolt circle. What is the center to center distance between any two holes?
3. Find the center to center distance x between the two holes shown, if they are spaced 26° apart.
4. A bronze casting 3′ 8″ in diameter is to have 6 holes drilled in it. If the bolt circle on which the holes are to be placed is 1½″ away from the outer edge of the casting, how far apart must the holes be placed?



Tapers. In designing a conical piece of work such as here shown, the amount of the slope of the sides is called the *taper*. Such a tapered piece is actually the frustum of a cone. The amount of taper is defined as the difference in diameter per unit length of the tapered part. Frequently only part of an entire piece is tapered, the rest of it being cylindrical. If D=large diameter, d=small diameter, L=length of tapered part only, T. T.=total taper, and T= the taper per inch, then, in the second diagram:

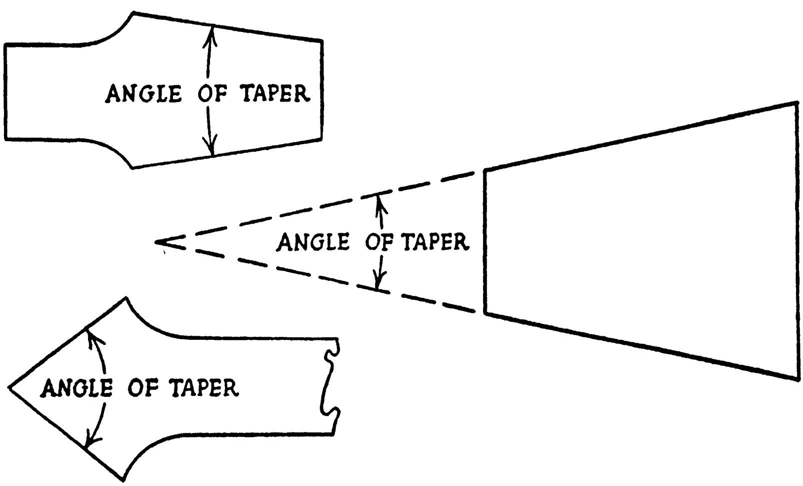




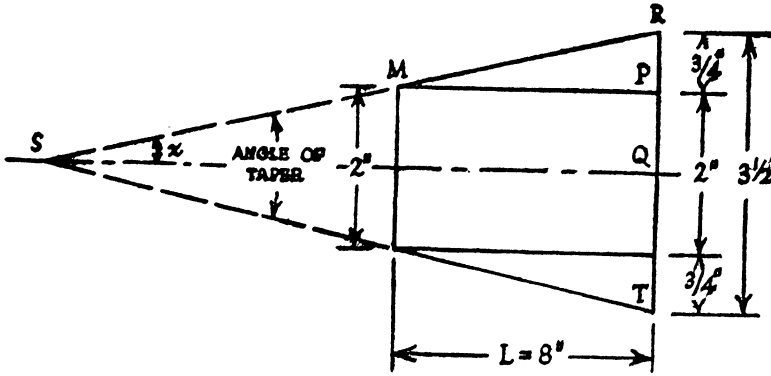
T. T.$=D-d=3''-2\frac{1}{2}''=\frac{1}{2}''$;

and $T=\dfrac{D-d}{L}=\dfrac{\frac{1}{2}}{5}=0.1''$ per inch.

Angle of Taper. The taper on a piece of work may be expressed as so many inches per inch, or as many inches per foot. It may also be described in terms of the angle included between the sides (or the prolongation of the sides) of the piece of work.



Note carefully that the angle between a sloping side (or its extension) and the center line represents only *half* the taper angle. Computation of tapers and taper angles involves the tangent of an angle, very simply.



It is clearly seen that △ RMP is similar to △ RSQ, and that ∠RMP= ∠RSQ=½ (taper angle). Hence:

$$\tan x = \frac{RP}{MP} = \frac{\frac{1}{2}(D-d)}{L} = \frac{\frac{1}{2}(3\frac{1}{2}-2)}{8} = \frac{\frac{3}{4}}{8} = .0938, \text{ or } x = 5°21\frac{1}{2}'.$$

taper angle$=2\times5°21\frac{1}{2}'=10°43'$.

EXAMPLE 1: What is the angle of taper in a piece of work having a taper of ¾″ per foot?

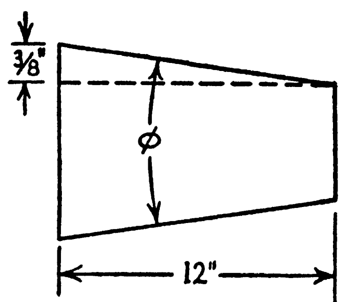
SOLUTION:

$$\tan(\tfrac{1}{2} \text{ taper angle}) = \frac{\frac{1}{2}(\text{taper})}{\text{length in inches}}$$

$$\tan\frac{\phi}{2} = \frac{\frac{1}{2}\times\frac{3}{4}}{12} = \tfrac{1}{2}\times\tfrac{3}{4}\times\tfrac{1}{12} = .0313.$$

$$\frac{\phi}{2} = 1°48'$$

$\phi=2\times1°48'=3°36'$, *Ans.*



EXAMPLE 2: What is the taper per foot of a piece of work if the angle of taper is 8°30′?

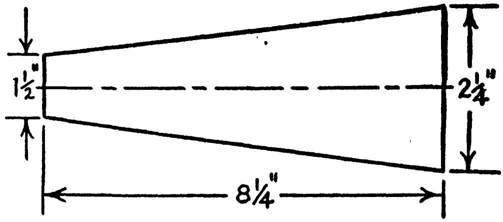
SOLUTION: $\frac{1}{2}(\text{taper angle})=\frac{1}{2}\times8°30'=4°15'$.
$\tan 4°15'=.0743$.

Hence taper per inch$=2\times.0743=.1486''$,
and taper per foot$=12\times.1486=1.7832''$, *Ans.*

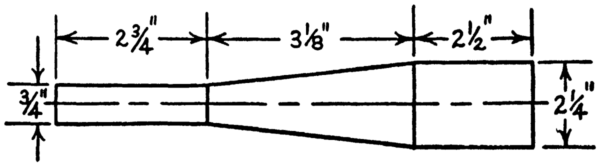
In other words, to find the taper per inch when the taper angle is given, we find the tangent of half the taper angle; this gives the taper from the center line in inches per inch. To find the total taper per inch, we multiply this result by 2. To find the total taper per foot, multiply the total taper per inch by 12.

Exercise 74.

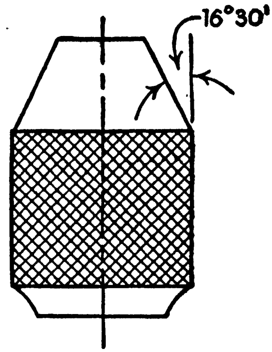
1. Find the angle of taper of the piece shown in the accompanying diagram.



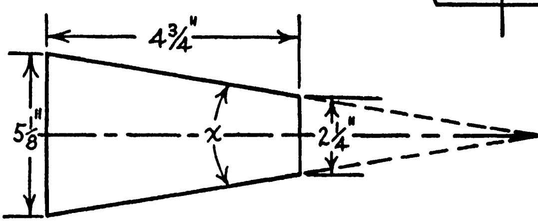
2. Find the taper angle in the following piece of work.



3. In the tapered drill chuck shown, find (a) the angle of taper, and (b) the taper per inch.

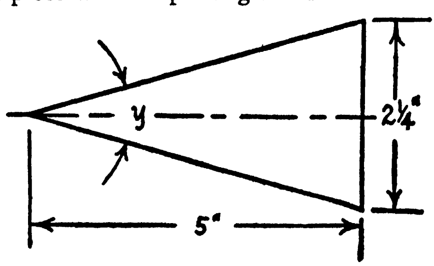


4. Determine angle x in the tapered piece shown.



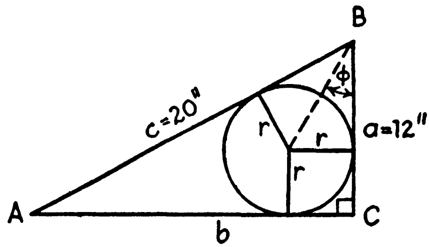
5. What is the taper per foot of a piece whose taper angle is 12°10′?

6. Find $\angle y$; if the length which is now 5″ were doubled, what would $\angle y$ then be equal to?



Miscellaneous Applications. Many problems of design in the machine shop are readily solved by making use of trigonometric relationships. Study the following and you will see how helpful trigonometry can be.

EXAMPLE 1: Find the diameter of the circle inscribed in a right triangle, one of whose sides is 12″ and whose hypotenuse is 20″; also, find ϕ.



SOLUTION: It can be shown that, when a circle is inscribed in any right triangle, the following relation holds:

hypotenuse+diameter=sum of the sides;

$c+2r=a+b$

or, diameter$=a+b-c$

Hence, $b=\sqrt{(20)^2-(12)^2}=16''$

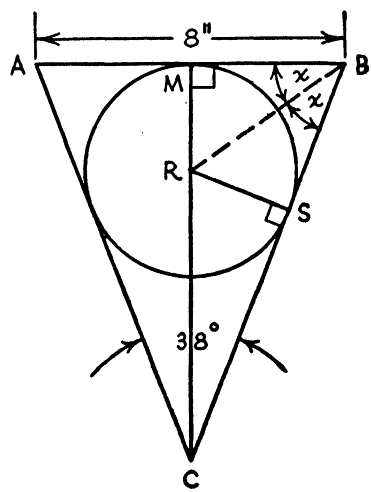
and diameter$=12+16-20=8''$, *Ans.*

$\tan B=16/12=1.3333$

$B=53°8'$

$\phi=\frac{1}{2}(B)=26°34'$, *Ans.*

EXAMPLE 2: Given a circle inscribed in an isosceles triangle ABC. Find the diameter of the circle if AB $=8''$ and $\angle ABC=38°$.



SOLUTION:

$\angle MCB=19°$; $MB=4''$;

$\angle x=\frac{1}{2}(90°-19°)=35°30'$

$MR=(MB)(\tan x)$

$=(4)(\tan 35°30')$

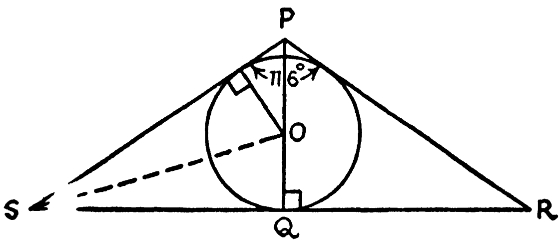
$=(4)(.7133)=2.8532$

diameter$=2\times 2.8532=5.706''$, *Ans.*

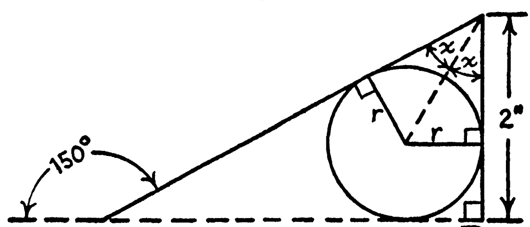
Exercise 75.

1. Find the diameter of a circle inscribed in a right triangle whose hypotenuse is 18″ and whose longer side is 10″.
2. What is the diameter of a circle inscribed in a right triangle whose shorter side is 6″, if the angle adjacent to that side is 42°?

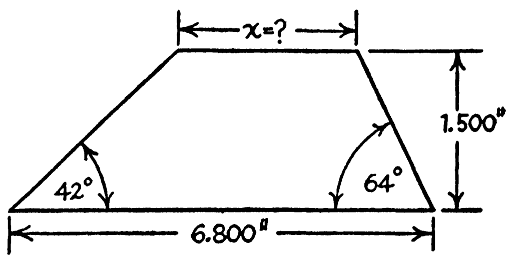
3. If PQ=12″, and ∠SPR=116°, find the radius of the inscribed circle.

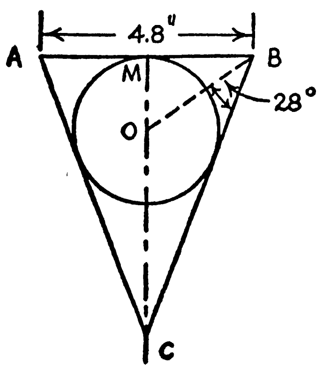


4. Find the diameter of the inscribed circle in the diagram shown.



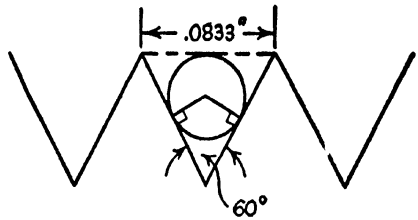
5. Find the shorter base of the trapezoid in the given diagram.



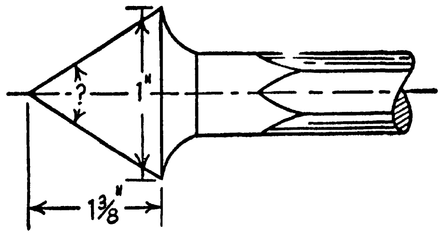


6. If ∠OBC=28°, find (a) the diameter of the inscribed circle, and (b) the depth MC.

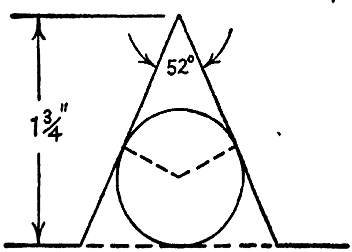
7. Find the diameter of the wire flush with the top of the 60°-screw thread shown, if the distance between two peaks is .0833″.

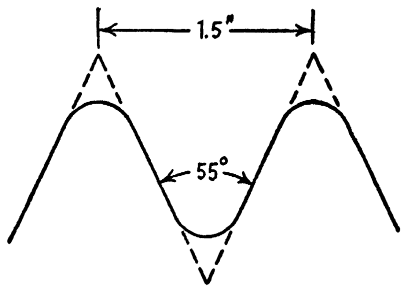


8. Find the included angle in the point of the countersink tool shown.

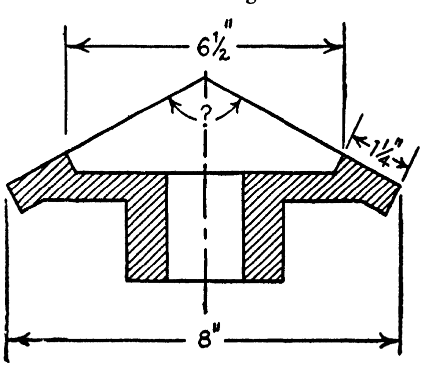


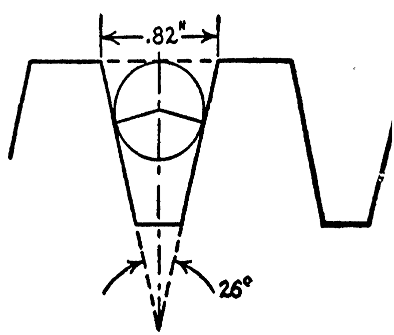
9. Find the diameter of the plug required to fit the 52° angle shown.





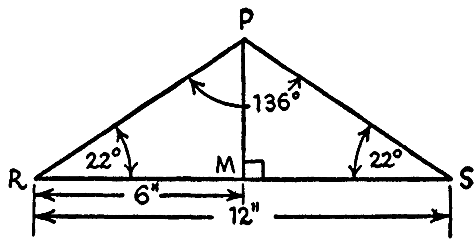
10. The special 55°-thread screw has a depth equal to ⅔ of the depth of the triangle formed when the sides of the thread are extended. Find the depth of the thread when the distance from top to top is 1.5″.
11. Find the included angle of the bevel gear blank shown below.
12. Find the diameter of the wire inserted in the worm thread with the 26°-included angle as shown.





22. SOLUTION OF OBLIQUE TRIANGLES

Oblique Triangles. By an oblique triangle is meant a triangle none of whose angles is a right angle. To solve an oblique triangle means to find the remaining sides and angles when some of these parts are known. This can sometimes be done by breaking the triangle down into right triangles; for example, if we wish to find the sides of an isosceles triangle whose base is 12″ and whose vertex angle in 136°, we draw PM perpendicular to RS, which divides the given triangle into two right triangles from which we can readily find PR. Thus:



$$\frac{RM}{RP} = \sin 68°$$

$$RP = \frac{RM}{\sin 68°} = \frac{6}{.9272} = 6.5, \textit{ Ans.}$$

However, it is not always convenient or possible to do this with oblique triangles, and so other methods must be used. We shall continue to use the standard notation for the sides and angles of a triangle, exactly as was done in the case of right triangles.

Functions of an Obtuse Angle. Before introducing these new methods, however, it is necessary to show how to find the functions of an obtuse angle. We already know how to find the functions of $(90°—a)$; thus

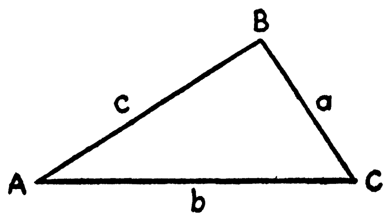
$$\sin (90°—a) = \cos a$$
$$\cos (90°—a) = \sin a$$
$$\tan (90°—a) = \cot a$$

It can be proved, although we shall not stop to do it here, that the transformations for finding functions of $(90°+a)$ and $(180°—a)$ are as follows:

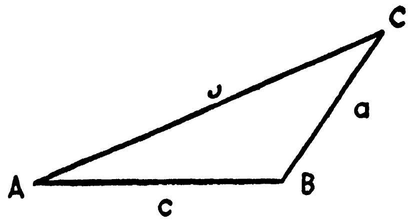
$$\sin (90°+a) = \cos a \qquad \sin (180°—a) = \sin a$$
$$\cos (90°+a) = —\sin a \qquad \cos (180°—a) = —\cos a$$
$$\tan (90°+a) = —\cot a \qquad \tan (180°—a) = —\tan a$$

Law of Sines. A convenient trigonometrical relationship which may be utilized in solving oblique triangles is the *law of sines,* which may be stated as follows:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



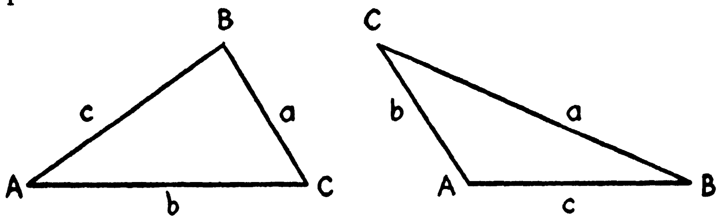
Expressed in words: *any side of a triangle is to the sine of the angle opposite that side as any other side is to the sine of its opposite angle.* Or, it may be written in another form, viz.:



$$\frac{a}{b}=\frac{\sin A}{\sin B};\quad \frac{a}{c}=\frac{\sin A}{\sin C};\quad \frac{b}{c}=\frac{\sin B}{\sin C}.$$

That is, the ratio of any two sides of a triangle is equal to the ratio of the sines of the angles opposite them, respectively.

Law of Cosines. This law states that in any triangle, the square of any side equals the sum of the squares of the other two sides diminished by the product of those two sides and the cosine of their included angle.



$$a^2=b^2+c^2-2bc\cos A$$
$$b^2=a^2+c^2-2ac\cos B$$
$$c^2=a^2+b^2-2ab\cos C.$$

Or, solving for the angles, we get:

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}$$

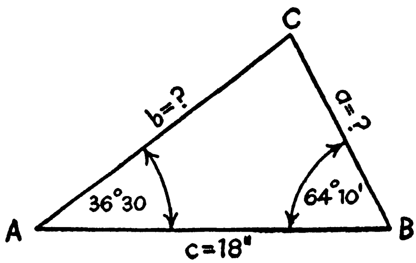
$$\cos C=\frac{a^2+b^2-c^2}{2ab}.$$

Types of Problems in Solving Oblique Triangles. It is convenient to consider four types of problems, or sets of given conditions, when dealing with oblique triangles, viz.:

I. Given one side and any two angles.
II. Given two sides and the included angle.
III. Given two sides and an angle opposite one of them.
IV. Given the three sides only.

We shall explain the method used to solve each of these four types; in the illustrative problems we shall use the tables in the back of the book.

TYPE I: Given One Side and Any Two Angles. In this case the two given angles may be adjacent to the given side (a.s.a.) or they may not be (s.a.a.); it makes no difference, however, since if any two angles of a triangle are known, the third may be found immediately by subtracting their sum from 180°.



EXAMPLE 1: Given: $C=18''$, $A=36°30'$, and $B=64°10'$. Find the remaining sides.

SOLUTION: $C=180°-(36°30'+64°10')=79°20'$

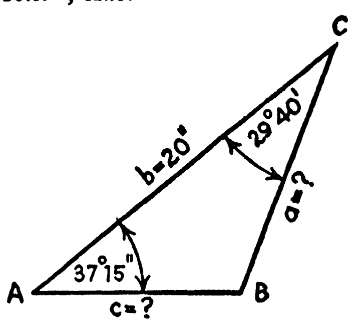
$$\frac{b}{c}=\frac{\sin B}{\sin C} \qquad \frac{a}{c}=\frac{\sin A}{\sin C}$$

$$\frac{b}{18}=\frac{\sin 64°10'}{\sin 79°20'} \qquad \frac{a}{18}=\frac{\sin 36°30'}{\sin 79°20'}$$

$$\frac{b}{18}=\frac{.9001}{.9827} \qquad \frac{a}{18}=\frac{.5948}{.9827}$$

$$b=\frac{(18)(.9001)}{.9827} \qquad a=\frac{(18)(.5948)}{.9827}$$

$b=16.49''$, *Ans.* $\qquad a=10.89''$, *Ans.*



EXAMPLE 2: Given: $b=20''$, $A=37°15'$, and $C=29°40'$. Solve the triangle.

SOLUTION: $B=180°-(37°15'+29°40')=113°5'$

$\sin B=\sin 113°5'=\sin(90°+23°5')=\cos 23°5'$

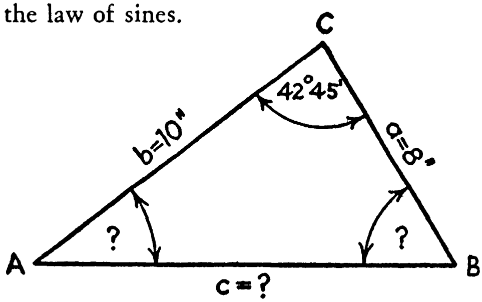
$$\frac{c}{b}=\frac{\sin C}{\sin B} \qquad \frac{a}{b}=\frac{\sin A}{\sin B}$$

$$\frac{c}{20}=\frac{\sin 29°40'}{\cos 23°5'} \qquad \frac{a}{20}=\frac{\sin 37°15'}{\cos 23°5'}$$

$$\frac{c}{20}=\frac{.4950}{.9199}$$
$$c=\frac{(20)(.4950)}{.9199}$$
$c=10.76''$, *Ans.*

$$\frac{a}{20}=\frac{.6053}{.9199}$$
$$a=\frac{(20)(.6053)}{.9199}$$
$a=13.16''$, *Ans.*

TYPE II: Given Two Sides and the Included Angle. In this case we use the law of cosines, as well as the law of sines.



EXAMPLE 1: Given: $a=8''$, $b=10''$, and $C=42°45'$. Solve the triangle.

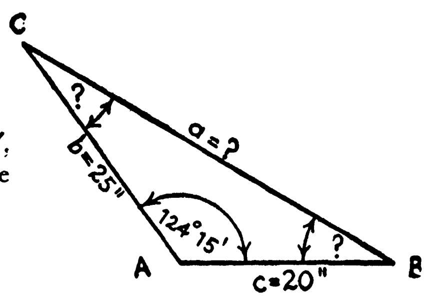
SOLUTION:
$$c^2=a^2+b^2-2ab\cos C$$
$$c^2=8^2+10^2-2(8)(10)(\cos 42°45')$$
$$c^2=64+100-(160)(.7343)=46.512$$
$c=\sqrt{45.87}=6.82''$, *Ans.*

Using the sine law to find A, we have:
$$\frac{a}{\sin A}=\frac{c}{\sin C}$$
$$\sin A=\frac{a\sin C}{c}=\frac{(8)(.6788)}{6.82}=.7962$$
$A=52°46'$, *Ans.*

Angle B may be found by difference:

$B=180°-(42°45'+52°46')=84°29'$, *Ans.*

As a check, use the law of sines:
$$\sin B=\frac{b\sin C}{c}=\frac{(10)(.6788)}{6.82}=.9953$$
from table, $\sin 84°29'=.9954$, Check.



EXAMPLE 2: Given: $b=25''$, $c=20''$, and $A=124°15'$. Solve the triangle.

SOLUTION: $a^2=b^2+c^2-2bc\cos A$

$$a^2=(25)^2+(20)^2-2(25)(20)(\cos 124°15').$$

But, $\cos 124°15'=\cos(90°+34°15')=-\sin 34°15'$

Hence, $a^2=625+400-(1000)(\sin 34°15')$

$$a^2=625+400-(1000)(.5628)=1587.8$$

$$a=\sqrt{1587.8}=39.847'', \textit{ Ans.}$$

$$\frac{c}{\sin C}=\frac{a}{\sin A}$$

$$\sin C=\frac{c\sin A}{a}=\frac{(20)(\sin 124°15')}{39.847}.$$

But, $\sin 124°15'=\sin(90°+34°15')=\cos 34°15'=.8266$

$$\sin C=\frac{(20)(.8266)}{39.847}=.4149$$

$C=24°31'$, *Ans.*

$B=180°-(124°15'+24°31')=31°12'$, *Ans.*

Check: $\sin B=\dfrac{b\sin A}{a}=\dfrac{(25)(.8266)}{39.847}=.5186$

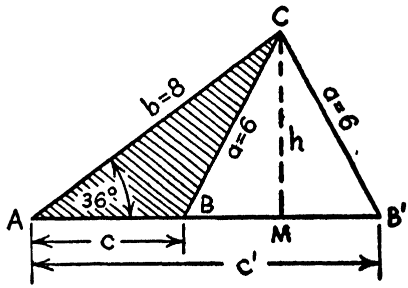
from the table, $\sin 31°12'=.5180$, Check.

Exercise 76.

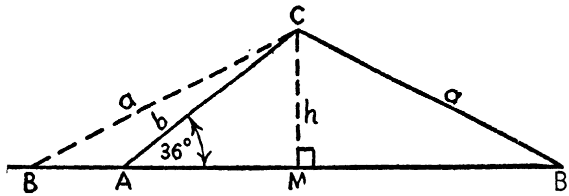
Solve the following oblique triangle, given the parts specified:

1. $A=42°30'$, $B=54°40'$, $c=20''$
2. $A=39°10'$, $C=112°$, $b=24''$
3. $A=38°45'$, $B=62°12'$, $a=18.2''$
4. $a=16''$, $c=10''$, $B=47°20'$
5. $a=20''$, $b=12''$, $C=135°$

TYPE III: Given Two Sides and an Angle Opposite One of Them. This case is a bit different from the others in that there is the possibility of two solutions, one solution, or no solution, depending upon the conditions of the problem. Consider the following diagram; given $a=6$, $b=8$, and $A=36°$. By studying the diagram it will be realized that not only the shaded triangle ABC, but also the large triangle AB′C (where CB=CB′) will satisfy the given conditions. In other words, if $a>h$ (or $b\sin A$) but $<b$, then *two* solutions are possible, viz.,

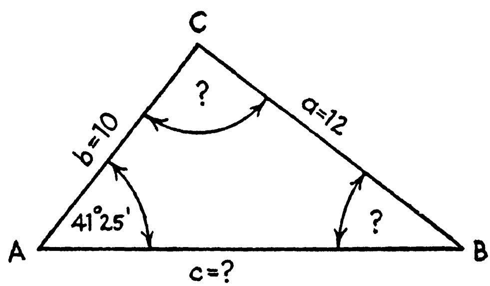


△ABC and △AB′C. If $a=h$ (or $b \sin A$), then only *one* solution is possible, viz., right triangle AMC. If $a<h$, *no* solution is possible, for no triangle exists in that event. Finally, if $a>h$ and also $a>b$, then again only *one* solution is possible, viz., △ACB′, since △ACB, while it contains the given parts a and b, does *not* contain ∠A (36°) as one of its interior angles, and hence does not represent a required solution. This will now be further illustrated.



EXAMPLE 1: Given: $a=12$, $b=10$, and $A=41°\,25'$. Solve the triangle.

SOLUTION: By constructing the triangle approximately, it is obvious that only one solution is possible.



$$\sin B=\frac{b \sin A}{a}=\frac{(10)(\sin 41°25')}{12}$$

$$\sin B=\frac{(10)(.6615)}{12}=.5513$$

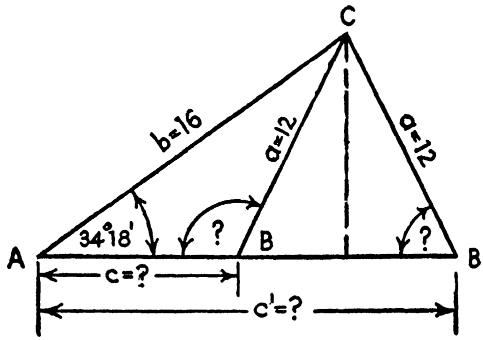
$B=33°27'$, *Ans.*

$C=180°-(41°25'+33°27')=105°8'$, *Ans.*

$$c=\frac{a \sin C}{\sin A}=\frac{(12)(\sin 105°8')}{\sin 41°25'}=\frac{(12)(\sin 74°52')}{\sin 41°25'}$$

$$c=\frac{(12)(.9653)}{.6615}=17.511, \textit{ Ans.}$$

EXAMPLE 2: Given: $a=12$, $b=16$, and $A=34°18'$. Solve the triangle.



SOLUTION: By constructing the triangle approximately, it is seen that two triangles are possible—either △ABC or △AB′C. We shall solve △AB′C first.

$$\sin B'=\frac{b \sin A}{a}=\frac{(16)(.5635)}{12}=.7180$$

$B'=45°54'$, *Ans.*

$C=180°-(A+B)=99°48'$, *Ans.*

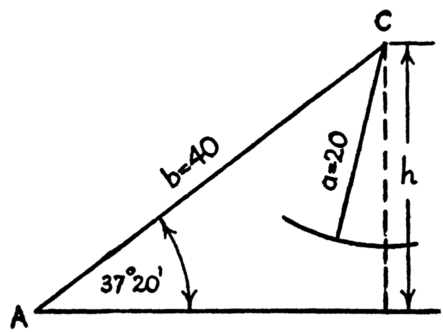
$$c'=\frac{a \sin C}{\sin A}=\frac{(12)(\sin 99°48')}{\sin 34°18'}=\frac{(12)(.9854)}{.5635}$$

$c'=20.984$, *Ans.*

Now, to find c in the *second* solution, we note that ∠CBA=180° —45°54′=134°6′, since △ CBB′ is isosceles and ∠CBB′=∠CB′B= 45°54′. Also note that ∠ABC therefore equals 11°36′.

Thus, $c=\frac{b \sin C}{\sin B}=\frac{(16)(\sin 11°36')}{\sin 134°6'}$; $c=\frac{(16)(.2011)}{.7181}=4.481$, *Ans.*

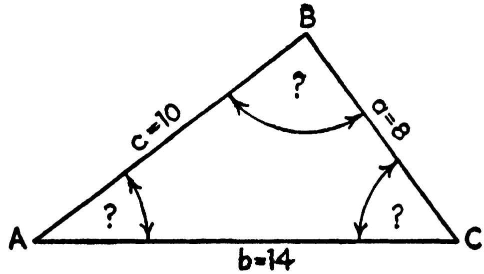
EXAMPLE 3: Given: $a=20$, $b=40$, and $A=37°20'$. Solve the triangle.



SOLUTION: In attempting to construct the triangle approximately, it will be seen that, since $h=b\ (\sin 37°20')=(40)(.6065)=24.26$, the length of a is $<h$ and so the triangle cannot be constructed with the parts given; thus no solution is possible.

TYPE IV: Given the Three Sides Only. When only the three sides and no angles are given, the law of cosines is used once more. In this case a triangle is always possible so long as the sum of any two of the given sides exceeds the third.

EXAMPLE: Given: $a=8$, $b=14$, $c=10$. Solve the triangle.



SOLUTION:

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{(14)^2+(10)^2-(8)^2}{2(14)\ (10)}=\frac{232}{280}=.8286$$

$A=34°3'$, *Ans.*

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{(8)^2+(14)^2-(10)^2}{2(8)(14)}=\frac{160}{224}=.7143$$

$C=44°25'$, *Ans.*

$B=180°-(A+C)=101°32'$, *Ans.*

Check:

$$\sin B=\frac{b\ \sin A}{a}=\frac{(14)(.5599)}{8}=.9798$$

$\sin 101°32'=\cos 11°32'=.9798$

Exercise 77.

Solve the triangles, given the following parts:

1. $A=66°$, $a=28''$, $b=16''$
2. $a=10''$, $b=12''$, $c=18''$
3. $b=5.6''$, $c=2.4''$, $B=110°$
4. $a=16.5''$, $b=12.8''$, $c=20''$
5. $A=27°30'$, $a=10''$, $c=18''$

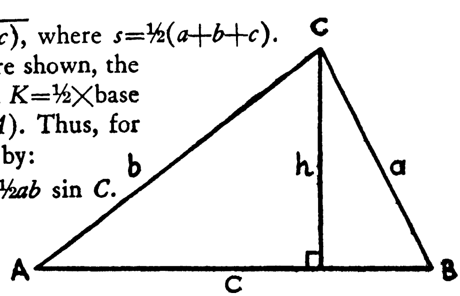
Area of Triangles. We have already seen, in the previous chapter, that the area of a triangle (K) is given by the following expressions:

(1) $K=\frac{1}{2}$ (base) (altitude);

(2) $K=\sqrt{s(s-a)(s-b)(s-c)}$, where $s=\frac{1}{2}(a+b+c)$.

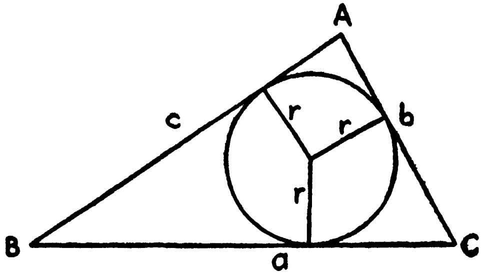
By considering the diagram here shown, the altitude $h=b\ \sin A$; hence area $K=\frac{1}{2}\times$base $\times$ altitude, or $K=\frac{1}{2}c(b\ \sin A)$. Thus, for any triangle, the area is given by:

$K=\frac{1}{2}bc\ \sin A=\frac{1}{2}ac\ \sin B=\frac{1}{2}ab\ \sin C.$



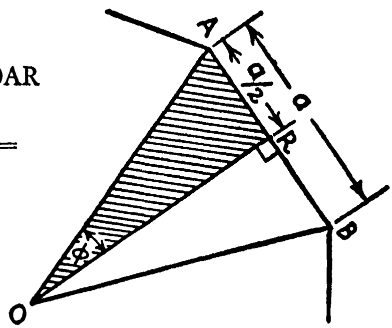
From (2) above, without actually proving it here, it can also be shown that the area of a triangle in terms of its sides and the radius of the inscribed circle is given by:

$K = s \times r$, where r=radius of the inscribed circle, and $s = \frac{1}{2}(a+b+c)$.



Area of Regular Polygons. If the side of a regular polygon is a, and there are n such sides, then $\phi = \frac{360°}{2n} = \frac{180°}{n}$, and OR=(AR) (cot ϕ) $= \frac{a}{2}$ (cot ϕ). Hence the area of $\triangle$OAR $= \frac{1}{2}(AR)(OR) = \frac{1}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)(\cot \phi) = \frac{a^2}{8}\cot \phi$. But as there are $2n$ such triangles in the entire regular polygon, the area of the polygon thus equals $2n \times \frac{a^2}{8}\cot \phi$, or

$$K = \frac{1}{4}na^2 \cot \frac{180°}{n}.$$



In a similar way it can be shown that the following relations also hold for regular polygons having n sides:

(1) Area of a regular polygon circumscribed about a circle whose radius is R:

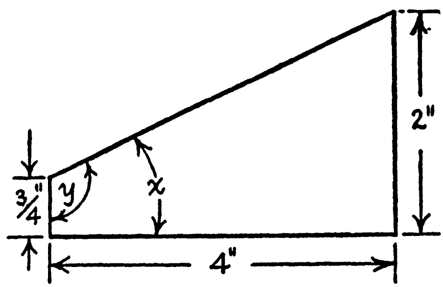
$$K = nR^2 \tan \frac{180°}{n}.$$

(2) Area of a regular polygon inscribed in a circle whose radius is r:

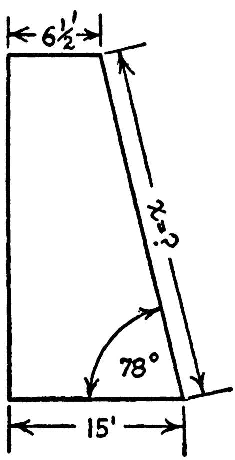
$$K = \frac{1}{2}nr^2 \sin \frac{360°}{n}.$$

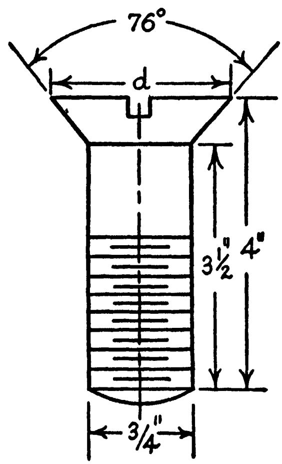
Exercise 78.

1. Find the area of an oblique triangle in which $A=34°30'$, $c=18.2''$, and $b=22.5''$.
2. Find the area of a parallelogram whose sides are 20″ and 32″, and one of whose angles is 42°20′.
3. Find the area of a regular polygon of 10 sides inscribed in a circle whose diameter is 40 cm.
4. Find the radius of a circle inscribed in a triangle whose sides are 12″, 18″ and 22″.
5. What is the area of a regular polygon of 12 sides, if each side of the polygon is 6″ long?
6. Find the number of degrees and minutes in angles x and y in the accompanying figure.



7. A parallelogram has two sides equal to 6″ and 9″, respectively, and the included angle is 39°. Find the shorter diagonal.
8. What is the length of the sloping side of the cross section of a concrete embankment if it makes an angle of 78° with the base, which is 15 ft. wide, and the top of the embankment is 6½ ft. wide?





9. Find the diameter d of the bolt with dimensions as shown.
10. The sides of a parallelogram are 10″ and 16″ long, respectively. If the longer diagonal is 20″, what are the angles of the parallelogram?

SELECTED REFERENCES FOR FURTHER STUDY

Trade Mathematics

AXELROD, A. Machine Shop Mathematics. McGraw-Hill Book Co.

BURNHAM, R. W. Mathematics for Machinists. J. Wiley & Sons.

CUSHMAN, F. Mathematics and the Machinist's Job. The Practical Mathematics of the Machinist's Trade. J. Wiley & Sons.

DOOLEY, W. H. & KRIEGEL, D. New Vocational Mathematics for Boys. D. C. Heath & Co.

FELKER, C. A. Shop Mathematics. Bruce Publishing Co.

HARPER, H. D. General Shop Mathematics. D. Van Nostrand Co.

JOHNSON, J. F. Applied Mathematics. Bruce Publishing Co.

MOYER, J. A. & SAMPSON, C. H. Practical Trade Mathematics. J. Wiley & Sons.

RAY, H. B. & DOUB, A. V. Preparatory Mathematics for Use in Technical Schools. J. Wiley & Sons.

SLADE, S. & MARGOLIS, L. Mathematics for Technical and Vocational Schools. J. Wiley & Sons.

Unit Course in Mathematics for Beginners in Machine Shop Practice. The Greenwood Co., Albany, N. Y.

VAN LEUVEN, E. P. General Trade Mathematics. McGraw-Hill Book Co.

WOLFE, J. H. & PHELPS, E. R. Practical Shop Mathematics: Vol. I, Elementary. McGraw-Hill Book Co.

′	0°		1°		2°		3°		4°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	0000	1.000	0175	9998	0349	9994	0523	9986	0698	9976	60
1	0003	1.000	0177	9998	0352	9994	0526	9986	0700	9975	59
2	0006	1.000	0180	9998	0355	9994	0529	9986	0703	9975	58
3	0009	1.000	0183	9998	0358	9994	0532	9986	0706	9975	57
4	0012	1.000	0186	9998	0361	9993	0535	9986	0709	9975	56
5	0015	1.000	0189	9998	0364	9993	0538	9986	0712	9975	55
6	0017	1.000	0192	9998	0366	9993	0541	9985	0715	9974	54
7	0020	1.000	0195	9998	0369	9993	0544	9985	0718	9974	53
8	0023	1.000	0198	9998	0372	9993	0547	9985	0721	9974	52
9	0026	1.000	0201	9998	0375	9993	0550	9985	0724	9974	51
10	0029	1.000	0204	9998	0378	9993	0552	9985	0727	9974	50
11	0032	1.000	0207	9998	0381	9993	0555	9985	0729	9973	49
12	0035	1.000	0209	9998	0384	9993	0558	9984	0732	9973	48
13	0038	1.000	0212	9998	0387	9993	0561	9984	0735	9973	47
14	0041	1.000	0215	9998	0390	9992	0564	9984	0738	9973	46
15	0044	1.000	0218	9998	0393	9992	0567	9984	0741	9973	45
16	0047	1.000	0221	9998	0396	9992	0570	9984	0744	9972	44
17	0049	1.000	0224	9997	0398	9992	0573	9984	0747	9972	43
18	0052	1.000	0227	9997	0401	9992	0576	9983	0750	9972	42
19	0055	1.000	0230	9997	0404	9992	0579	9983	0753	9972	41
20	0058	1.000	0233	9997	0407	9992	0581	9983	0756	9971	40
21	0061	1.000	0236	9997	0410	9992	0584	9983	0758	9971	39
22	0064	1.000	0239	9997	0413	9991	0587	9983	0761	9971	38
23	0067	1.000	0241	9997	0416	9991	0590	9983	0764	9971	37
24	0070	1.000	0244	9997	0419	9991	0593	9982	0767	9971	36
25	0073	1.000	0247	9997	0422	9991	0596	9982	0770	9970	35
26	0076	1.000	0250	9997	0425	9991	0599	9982	0773	9970	34
27	0079	1.000	0253	9997	0427	9991	0602	9982	0776	9970	33
28	0081	1.000	0256	9997	0430	9991	0605	9982	0779	9970	32
29	0084	1.000	0259	9997	0433	9991	0608	9982	0782	9969	31
30	0087	1.000	0262	9997	0436	9990	0610	9981	0785	9969	30
31	0090	1.000	0265	9996	0439	9990	0613	9981	0787	9969	29
32	0093	1.000	0268	9996	0442	9990	0616	9981	0790	9969	28
33	0096	1.000	0270	9996	0445	9990	0619	9981	0793	9968	27
34	0099	1.000	0273	9996	0448	9990	0622	9981	0796	9968	26
35	0102	9999	0276	9996	0451	9990	0625	9980	0799	9968	25
36	0105	9999	0279	9996	0454	9990	0628	9980	0802	9968	24
37	0108	9999	0282	9996	0457	9990	0631	9980	0805	9968	23
38	0111	9999	0285	9996	0459	9989	0634	9980	0808	9967	22
39	0113	9999	0288	9996	0462	9989	0637	9980	0811	9967	21
40	0116	9999	0291	9996	0465	9989	0640	9980	0814	9967	20
41	0119	9999	0294	9996	0468	9989	0642	9979	0816	9967	19
42	0122	9999	0297	9996	0471	9989	0645	9979	0819	9966	18
43	0125	9999	0300	9996	0474	9989	0648	9979	0822	9966	17
44	0128	9999	0302	9995	0477	9989	0651	9979	0825	9966	16
45	0131	9999	0305	9995	0480	9988	0654	9979	0828	9966	15
46	0134	9999	0308	9995	0483	9988	0657	9978	0831	9965	14
47	0137	9999	0311	9995	0486	9988	0660	9978	0834	9965	13
48	0140	9999	0314	9995	0488	9988	0663	9978	0837	9965	12
49	0143	9999	0317	9995	0491	9988	0666	9978	0840	9965	11
50	0145	9999	0320	9995	0494	9988	0669	9978	0843	9964	10
51	0148	9999	0323	9995	0497	9988	0671	9977	0845	9964	9
52	0151	9999	0326	9995	0500	9987	0674	9977	0848	9964	8
53	0154	9999	0329	9995	0503	9987	0677	9977	0851	9964	7
54	0157	9999	0332	9995	0506	9987	0680	9977	0854	9963	6
55	0160	9999	0334	9994	0509	9987	0683	9977	0857	9963	5
56	0163	9999	0337	9994	0512	9987	0686	9976	0860	9963	4
57	0166	9999	0340	9994	0515	9987	0689	9976	0863	9963	3
58	0169	9999	0343	9994	0518	9987	0692	9976	0866	9962	2
59	0172	9999	0346	9994	0520	9986	0695	9976	0869	9962	1
60	0175	9999	0349	9994	0523	9986	0698	9976	0872	9962	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	89°		88°		87°		86°		85°		′

′	5°		6°		7°		8°		9°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	0872	9962	1045	9945	1219	9925	1392	9903	1564	9877	60
1	0874	9962	1048	9945	1222	9925	1395	9902	1567	9876	59
2	0877	9961	1051	9945	1224	9925	1397	9902	1570	9876	58
3	0880	9961	1054	9944	1227	9924	1400	9901	1573	9876	57
4	0883	9961	1057	9944	1230	9924	1403	9901	1576	9875	56
5	0886	9961	1060	9944	1233	9924	1406	9901	1579	9875	55
6	0889	9960	1063	9943	1236	9923	1409	9900	1582	9874	54
7	0892	9960	1066	9943	1239	9923	1412	9900	1584	9874	53
8	0895	9960	1068	9943	1241	9923	1415	9899	1587	9873	52
9	0898	9960	1071	9942	1245	9922	1418	9899	1590	9873	51
10	0901	9959	1074	9942	1248	9922	1421	9899	1593	9872	50
11	0903	9959	1077	9942	1250	9922	1423	9898	1596	9872	49
12	0906	9959	1080	9942	1253	9921	1426	9898	1599	9871	48
13	0909	9959	1083	9941	1256	9921	1429	9897	1602	9871	47
14	0912	9958	1086	9941	1259	9920	1432	9897	1605	9870	46
15	0915	9958	1089	9941	1262	9920	1435	9897	1607	9870	45
16	0918	9958	1092	9940	1265	9920	1438	9896	1610	9869	44
17	0921	9958	1094	9940	1268	9919	1441	9896	1613	9869	43
18	0924	9957	1097	9940	1271	9919	1444	9895	1616	9869	42
19	0927	9957	1100	9939	1274	9919	1446	9895	1619	9868	41
20	0929	9957	1103	9939	1276	9918	1449	9894	1622	9868	40
21	0932	9956	1106	9939	1279	9918	1452	9894	1625	9867	39
22	0935	9956	1109	9938	1282	9917	1455	9894	1628	9867	38
23	0938	9956	1112	9938	1285	9917	1458	9893	1630	9866	37
24	0941	9956	1115	9938	1288	9917	1461	9893	1633	9866	36
25	0944	9955	1118	9937	1291	9916	1464	9802	1636	9865	35
26	0947	9955	1120	9937	1294	9916	1467	9892	1639	9865	34
27	0950	9955	1123	9937	1297	9916	1469	9891	1642	9864	33
28	0953	9955	1126	9936	1299	9915	1472	9891	1645	9864	32
29	0956	9954	1129	9936	1302	9915	1475	9891	1648	9863	31
30	0958	9954	1132	9936	1305	9914	1478	9890	1650	9863	30
31	0961	9954	1135	9935	1308	9914	1481	9890	1653	9862	29
32	0964	9953	1138	9935	1311	9914	1484	9889	1656	986[illegible]	28
33	0967	9953	1141	9935	1314	9913	1487	9889	1659	986[illegible]	27
34	0970	9953	1144	9934	1317	9913	1490	9888	1662	9861	26
35	0973	9953	1146	9934	1320	9913	1492	9888	1665	9860	25
36	0976	9952	1149	9934	1323	9912	1495	9888	1668	9860	24
37	0979	9952	1152	9933	1325	9912	1498	9887	1671	9859	23
38	0982	9952	1155	9933	1328	9911	1501	9887	1673	9859	22
39	0985	9951	1158	9933	1331	9911	1504	9886	1676	9859	21
40	0987	9951	1161	9932	1334	9911	1507	9886	1679	9858	20
41	0990	9951	1164	9932	1337	9910	1510	9885	1682	9858	19
42	0993	9951	1167	9932	1340	9910	1513	9885	1685	9857	18
43	0996	9950	1170	9931	1343	9909	1515	9884	1688	9857	17
44	0999	9950	1172	9931	1346	9909	1518	9884	1691	9856	16
45	1002	9950	1175	9931	1349	9909	1521	9884	1693	9856	15
46	1005	9949	1178	9930	1351	9908	1524	9883	1696	9855	14
47	1008	9949	1181	9930	1354	9908	1527	9883	1699	9855	13
48	1011	9949	1184	9930	1357	9907	1530	9882	1702	9854	12
49	1013	9949	1187	9929	1360	9907	1533	9882	1705	9854	11
50	1016	9948	1190	9929	1363	9907	1536	9881	1708	9853	10
51	1019	9948	1193	9929	1366	9906	1538	9881	1711	9853	9
52	1022	9948	1196	9928	1369	9906	1541	9880	1714	9852	8
53	1025	9947	1198	9928	1372	9905	1544	9880	1716	9852	7
54	1028	9947	1201	9928	1374	9905	1547	9880	1719	9851	6
55	1031	9947	1204	9927	1377	9905	1550	9879	1722	9851	5
56	1034	9946	1207	9927	1380	9904	1553	9879	1725	9850	4
57	1037	9946	1210	9927	1383	9904	1556	9878	1728	9850	3
58	1039	9946	1213	9926	1386	9903	1559	9878	1731	9849	2
59	1042	9946	1216	9926	1389	9903	1561	9877	1734	9849	1
60	1045	9945	1219	9925	1392	9903	1564	9877	1736	9848	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	84°		83°		82°		81°		80°		′

′	10°		11°		12°		13°		14°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	1736	9848	1908	9816	2079	9781	2250	9744	2419	9703	60
1	1739	9848	1911	9816	2082	9781	2252	9743	2422	9702	59
2	1742	9847	1914	9815	2085	9780	2255	9742	2425	9702	58
3	1745	9847	1917	9815	2088	9780	2258	9742	2428	9701	57
4	1748	9846	1920	9814	2090	9779	2261	9741	2431	9700	56
5	1751	9846	1922	9813	2093	9778	2264	9740	2433	9699	55
6	1754	9845	1925	9813	2096	9778	2267	9740	2436	9699	54
7	1757	9845	1928	9812	2099	9777	2269	9739	2439	9698	53
8	1759	9844	1931	9812	2102	9777	2272	9738	2442	9697	52
9	1762	9843	1934	9811	2105	9776	2275	9738	2445	9697	51
10	1765	9843	1937	9811	2108	9775	2278	9737	2447	9696	50
11	1768	9842	1939	9810	2110	9775	2281	9736	2450	9695	49
12	1771	9842	1942	9810	2113	9774	2284	9736	2453	9694	48
13	1774	9841	1945	9809	2116	9774	2286	9735	2456	9694	47
14	1777	9841	1948	9808	2119	9773	2289	9734	2459	9693	46
15	1779	9840	1951	9808	2122	9772	2292	9734	2462	9692	45
16	1782	9840	1954	9807	2125	9772	2295	9733	2464	9692	44
17	1785	9839	1957	9807	2127	9771	2298	9732	2467	9691	43
18	1788	9839	1959	9806	2130	9770	2300	9732	2470	9690	42
19	1791	9838	1962	9806	2133	9770	2303	9731	2473	9689	41
20	1794	9838	1965	9805	2136	9769	2306	9730	2476	9689	40
21	1797	9837	1968	9804	2139	9769	2309	9730	2478	9688	39
22	1799	9837	1971	9804	2142	9768	2312	9729	2481	9687	38
23	1802	9836	1974	9803	2145	9767	2315	9728	2484	9687	37
24	1805	9836	1977	9803	2147	9767	2317	9728	2487	9686	36
25	1808	9835	1979	9802	2150	9766	2320	9727	2490	9685	35
26	1811	9835	1982	9802	2153	9765	2323	9726	2493	9684	34
27	1814	9834	1985	9801	2156	9765	2326	9726	2495	9684	33
28	1817	9834	1988	9800	2159	9764	2329	9725	2498	9683	32
29	1819	9833	1991	9800	2162	9764	2332	9724	2501	9682	31
30	1822	9833	1994	9799	2164	9763	2334	9724	2504	9681	30
31	1825	9832	1997	9799	2167	9762	2337	9723	2507	9681	29
32	1828	9831	1999	9798	2170	9762	2340	9722	2509	9680	28
33	1831	9831	2002	9798	2173	9761	2343	9722	2512	9679	27
34	1834	9830	2005	9797	2176	9760	2346	9721	2515	9679	26
35	1837	9830	2008	9796	2179	9760	2349	9720	2518	9678	25
36	1840	9829	2011	9796	2181	9759	2351	9720	2521	9677	24
37	1842	9829	2014	9795	2184	9759	2354	9719	2524	9676	23
38	1845	9828	2016	9795	2187	9758	2357	9718	2526	9676	22
39	1848	9828	2019	9794	2190	9757	2360	9718	2529	9675	21
40	1851	9827	2022	9793	2193	9757	2363	9717	2532	9674	20
41	1854	9827	2025	9793	2196	9756	2366	9716	2535	9673	19
42	1857	9826	2028	9792	2198	9755	2368	9715	2538	9673	18
43	1860	9826	2031	9792	2201	9755	2371	9715	2540	9672	17
44	1862	9825	2034	9791	2204	9754	2374	9714	2543	9671	16
45	1865	9825	2036	9790	2207	9753	2377	9713	2546	9670	15
46	1868	9824	2039	9790	2210	9753	2380	9713	2549	9670	14
47	1871	9823	2042	9789	2213	9752	2383	9712	2552	9669	13
48	1874	9823	2045	9789	2215	9751	2385	9711	2554	9668	12
49	1877	9822	2048	9788	2218	9751	2388	9711	2557	9667	11
50	1880	9822	2051	9787	2221	9750	2391	9710	2560	9667	10
51	1882	9821	2054	9787	2224	9750	2394	9709	2563	9666	9
52	1885	9821	2056	9786	2227	9749	2397	9709	2566	9665	8
53	1888	9820	2059	9786	2230	9748	2399	9708	2569	9665	7
54	1891	9820	2062	9785	2233	9748	2402	9707	2571	9664	6
55	1894	9819	2065	9784	2235	9747	2405	9706	2574	9663	5
56	1897	9818	2068	9784	2238	9746	2408	9706	2577	9662	4
57	1900	9818	2071	9783	2241	9746	2411	9705	2580	9662	3
58	1902	9817	2073	9783	2244	9745	2414	9704	2583	9661	2
59	1905	9817	2076	9782	2247	9744	2416	9704	2585	9660	1
60	1908	9816	2079	9781	2250	9744	2419	9703	2588	9659	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	79°		78°		77°		76°		75°		′

′	15°		16°		17°		18°		19°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	2588	9659	2756	9613	2924	9563	3090	9511	3256	9455	60
1	2591	9659	2759	9612	2926	9562	3093	9510	3258	9454	59
2	2594	9658	2762	9611	2929	9561	3096	9509	3261	9453	58
3	2597	9657	2765	9610	2932	9560	3098	9508	3264	9452	57
4	2599	9656	2768	9609	2935	9560	3101	9507	3267	9451	56
5	2602	9655	2770	9609	2938	9559	3104	9506	3269	9450	55
6	2605	9655	2773	9608	2940	9558	3107	9505	3272	9449	54
7	2608	9654	2776	9607	2943	9557	3110	9504	3275	9449	53
8	2611	9653	2779	9606	2946	9556	3112	9503	3278	9448	52
9	2613	9652	2782	9605	2949	9555	3115	9502	3280	9447	51
10	2616	9652	2784	9605	2952	9555	3118	9502	3283	9446	50
11	2619	9651	2787	9604	2954	9554	3121	9501	3286	9445	49
12	2622	9650	2790	9603	2957	9553	3123	9500	3289	9444	48
13	2625	9649	2793	9602	2960	9552	3126	9499	3291	9443	47
14	2628	9649	2795	9601	2963	9551	3129	9498	3294	9442	46
15	2630	9648	2798	9600	2965	9550	3132	9497	3297	9441	45
16	2633	9647	2801	9600	2968	9549	3134	9496	3300	9440	44
17	2636	9646	2804	9599	2971	9548	3137	9495	3302	9439	43
18	2639	9646	2807	9598	2974	9548	3140	9494	3305	9438	42
19	2642	9645	2809	9597	2977	9547	3143	9493	3308	9437	41
20	2644	9644	2812	9596	2979	9546	3145	9492	3311	9436	40
21	2647	9643	2815	9596	2982	9545	3148	9492	3313	9435	39
22	2650	9642	2818	9595	2985	9544	3151	9491	3316	9434	38
23	2653	9642	2821	9594	2988	9543	3154	9490	3319	9433	37
24	2656	9641	2823	9593	2990	9542	3156	9489	3322	9432	36
25	2658	9640	2826	9592	2993	9542	3159	9488	3324	9431	35
26	2661	9639	2829	9591	2996	9541	3162	9487	3327	9430	34
27	2664	9639	2832	9591	2999	9540	3165	9486	3330	9429	33
28	2667	9638	2835	9590	3002	9539	3168	9485	3333	9428	32
29	2670	9637	2837	9589	3004	9538	3170	9484	3335	9427	31
30	2672	9636	2840	9588	3007	9537	3173	9483	3338	9426	30
31	2675	9636	2843	9587	3010	9536	3176	9482	3341	9425	29
32	2678	9635	2846	9587	3013	9535	3179	9481	3344	9424	28
33	2681	9634	2849	9586	3015	9535	3181	9480	3346	9423	27
34	2684	9633	2851	9585	3018	9534	3184	9480	3349	9423	26
35	2686	9632	2854	9584	3021	9533	3187	9479	3352	9422	25
36	2689	9632	2857	9583	3024	9532	3190	9478	3355	9421	24
37	2692	9631	2860	9582	3026	9531	3192	9477	3357	9420	23
38	2695	9630	2862	9582	3029	9530	3195	9476	3360	9419	22
39	2698	9629	2865	9581	3032	9529	3198	9475	3363	9418	21
40	2700	9628	2868	9580	3035	9528	3201	9474	3365	9417	20
41	2703	9628	2871	9579	3038	9527	3203	9473	3368	9416	19
42	2706	9627	2874	9578	3040	9527	3206	9472	3371	9415	18
43	2709	9626	2876	9577	3043	9526	3209	9471	3374	9414	17
44	2712	9625	2879	9577	3046	9525	3212	9470	3376	9413	16
45	2714	9625	2882	9576	3049	9524	3214	9469	3379	9412	15
46	2717	9624	2885	9575	3051	9523	3217	9468	3382	9411	14
47	2720	9623	2888	9574	3054	9522	3220	9467	3385	9410	13
48	2723	9622	2890	9573	3057	9521	3223	9466	3387	9409	12
49	2726	9621	2893	9572	3060	9520	3225	9466	3390	9408	11
50	2728	9621	2896	9572	3062	9520	3228	9465	3393	9407	10
51	2731	9620	2899	9571	3065	9519	3231	9464	3396	9406	9
52	2734	9619	2901	9570	3068	9518	3234	9463	3398	9405	8
53	2737	9618	2904	9569	3071	9517	3236	9462	3401	9404	7
54	2740	9617	2907	9568	3074	9516	3239	9461	3404	9403	6
55	2742	9617	2910	9567	3076	9515	3242	9460	3407	9402	5
56	2745	9616	2913	9566	3079	9514	3245	9459	3409	9401	4
57	2748	9615	2915	9566	3082	9513	3247	9458	3412	9400	3
58	2751	9614	2918	9565	3085	9512	3250	9457	3415	9399	2
59	2754	9613	2921	9564	3087	9511	3253	9456	3417	9398	1
60	2756	9613	2924	9563	3090	9511	3256	9455	3420	9397	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	74°		73°		72°		71°		70°		′

′	20° sin	20° cos	21° sin	21° cos	22° sin	22° cos	23° sin	23° cos	24° sin	24° cos	′
0	3420	9397	3584	9336	3746	9272	3907	9205	4067	9135	60
1	3423	9396	3586	9335	3749	9271	3910	9204	4070	9134	59
2	3426	9395	3589	9334	3751	9270	3913	9203	4073	9133	58
3	3428	9394	3592	9333	3754	9269	3915	9202	4075	9132	57
4	3431	9393	3595	9332	3757	9267	3918	9200	4078	9131	56
5	3434	9392	3597	9331	3760	9266	3921	9199	4081	9130	55
6	3437	9391	3600	9330	3762	9265	3923	9198	4083	9128	54
7	3439	9390	3603	9328	3765	9264	3926	9197	4086	9127	53
8	3442	9389	3605	9327	3768	9263	3929	9196	4089	9126	52
9	3445	9388	3608	9326	3770	9262	3931	9195	4091	9125	51
10	3448	9387	3611	9325	3773	9261	3934	9194	4094	9124	50
11	3450	9386	3614	9324	3776	9260	3937	9192	4097	9122	49
12	3453	9385	3616	9323	3778	9259	3939	9191	4099	9121	48
13	3456	9384	3619	9322	3781	9258	3942	9190	4102	9120	47
14	3458	9383	3622	9321	3784	9257	3945	9189	4105	9119	46
15	3461	9382	3624	9320	3786	9255	3947	9188	4107	9118	45
16	3464	9381	3627	9319	3789	9254	3950	9187	4110	9116	44
17	3467	9380	3630	9318	3792	9253	3953	9186	4112	9115	43
18	3469	9379	3633	9317	3795	9252	3955	9184	4115	9114	42
19	3472	9378	3635	9316	3797	9251	3958	9183	4118	9113	41
20	3475	9377	3638	9315	3800	9250	3961	9182	4120	9112	40
21	3478	9376	3641	9314	3803	9249	3963	9181	4123	9110	39
22	3480	9375	3643	9313	3805	9248	3966	9180	4126	9109	38
23	3483	9374	3646	9312	3808	9247	3969	9179	4128	9108	37
24	3486	9373	3649	9311	3811	9245	3971	9178	4131	9107	36
25	3488	9372	3651	9309	3813	9244	3974	9176	4134	9106	35
26	3491	9371	3654	9308	3816	9243	3977	9175	4136	9104	34
27	3494	9370	3657	9307	3819	9242	3979	9174	4139	9103	33
28	3497	9369	3660	9306	3821	9241	3982	9173	4142	9102	32
29	3499	9368	3662	9305	3824	9240	3985	9172	4144	9101	31
30	3502	9367	3665	9304	3827	9239	3987	9171	4147	9100	30
31	3505	9366	3668	9303	3830	9238	3990	9169	4150	9098	29
32	3508	9365	3670	9302	3832	9237	3993	9168	4152	9097	28
33	3510	9364	3673	9301	3835	9235	3995	9167	4155	9096	27
34	3513	9363	3676	9300	3838	9234	3998	9166	4158	9095	26
35	3516	9362	3679	9299	3840	9233	4001	9165	4160	9094	25
36	3518	9361	3681	9298	3843	9232	4003	9164	4163	9092	24
37	3521	9360	3684	9297	3846	9231	4006	9162	4165	9091	23
38	3524	9359	3687	9296	3848	9230	4009	9161	4168	9090	22
39	3527	9358	3689	9295	3851	9229	4011	9160	4171	9089	21
40	3529	9356	3692	9293	3854	9228	4014	9159	4173	9088	20
41	3532	9355	3695	9292	3856	9227	4017	9158	4176	9086	19
42	3535	9354	3697	9291	3859	9225	4019	9157	4179	9085	18
43	3537	9353	3700	9290	3862	9224	4022	9155	4181	9084	17
44	3540	9352	3703	9289	3864	9223	4025	9154	4184	9083	16
45	3543	9351	3706	9288	3867	9222	4027	9153	4187	9081	15
46	3546	9350	3708	9287	3870	9221	4030	9152	4189	9080	14
47	3548	9349	3711	9286	3872	9220	4033	9151	4192	9079	13
48	3551	9348	3714	9285	3875	9219	4035	9150	4195	9078	12
49	3554	9347	3716	9284	3878	9218	4038	9148	4197	9077	11
50	3557	9346	3719	9283	3881	9216	4041	9147	4200	9075	10
51	3559	9345	3722	9282	3883	9215	4043	9146	4202	9074	9
52	3562	9344	3724	9281	3886	9214	4046	9145	4205	9073	8
53	3565	9343	3727	9279	3889	9213	4049	9144	4208	9072	7
54	3567	9342	3730	9278	3891	9212	4051	9143	4210	9070	6
55	3570	9341	3733	9277	3894	9211	4054	9141	4213	9069	5
56	3573	9340	3735	9276	3897	9210	4057	9140	4216	9068	4
57	3576	9339	3738	9275	3899	9208	4059	9139	4218	9067	3
58	3578	9338	3741	9274	3902	9207	4062	9138	4221	9066	2
59	3581	9337	3743	9273	3905	9206	4065	9137	4224	9064	1
60	3584	9336	3746	9272	3907	9205	4067	9135	4226	9063	0
′	cos 69°	sin 69°	cos 68°	sin 68°	cos 67°	sin 67°	cos 66°	sin 66°	cos 65°	sin 65°	′

′	25°		26°		27°		28°		29°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	4226	9063	4384	8988	4540	8910	4695	8829	4848	8746	60
1	4229	9062	4386	8987	4542	8909	4697	8828	4851	8745	59
2	4231	9061	4389	8985	4545	8907	4700	8827	4853	8743	58
3	4234	9059	4392	8984	4548	8906	4702	8825	4856	8742	57
4	4237	9058	4394	8983	4550	8905	4705	8824	4858	8741	56
5	4239	9057	4397	8982	4553	8903	4708	8823	4861	8739	55
6	4242	9056	4399	8980	4555	8902	4710	8821	4863	8738	54
7	4245	9054	4402	8979	4558	8901	4713	8820	4866	8736	53
8	4247	9053	4405	8978	4561	8899	4715	8819	4868	8735	52
9	4250	9052	4407	8976	4563	8898	4718	8817	4871	8733	51
10	4253	9051	4410	8975	4566	8897	4720	8816	4874	8732	50
11	4255	9050	4412	8974	4568	8895	4723	8814	4876	8731	49
12	4258	9048	4415	8973	4571	8894	4726	8813	4879	8729	48
13	4260	9047	4418	8971	4574	8893	4728	8812	4881	8728	47
14	4263	9046	4420	8970	4576	8892	4731	8810	4884	8726	46
15	4266	9045	4423	8969	4579	8890	4733	8809	4886	8725	45
16	4268	9043	4425	8967	4581	8889	4736	8808	4889	8724	44
17	4271	9042	4428	8966	4584	8888	4738	8806	4891	8722	43
18	4274	9041	4431	8965	4586	8886	4741	8805	4894	8721	42
19	4276	9040	4433	8964	4589	8885	4743	8803	4896	8719	41
20	4279	9038	4436	8962	4592	8884	4746	8802	4899	8718	40
21	4281	9037	4439	8961	4594	8882	4749	8801	4901	8716	39
22	4284	9036	4441	8960	4597	8881	4751	8799	4904	8715	38
23	4287	9035	4444	8958	4599	8879	4754	8798	4907	8714	37
24	4289	9033	4446	8957	4602	8878	4756	8796	4909	8712	36
25	4292	9032	4449	8956	4605	8877	4759	8795	4912	8711	35
26	4295	9031	4452	8955	4607	8875	4761	8794	4914	8709	34
27	4297	9030	4454	8953	4610	8874	4764	8792	4917	8708	33
28	4300	9028	4457	8952	4612	8873	4766	8791	4919	8706	32
29	4302	9027	4459	8951	4615	8871	4769	8790	4922	8705	31
30	4305	9026	4462	8949	4617	8870	4772	8788	4924	8704	30
31	4308	9025	4465	8948	4620	8869	4774	8787	4927	8702	29
32	4310	9023	4467	8947	4623	8867	4777	8785	4929	8701	28
33	4313	9022	4470	8945	4625	8866	4779	8784	4932	8699	27
34	4316	9021	4472	8944	4628	8865	4782	8783	4934	8698	26
35	4318	9020	4475	8943	4630	8863	4784	8781	4937	8696	25
36	4321	9018	4478	8942	4633	8862	4787	8780	4939	8695	24
37	4323	9017	4480	8940	4636	8861	4789	8778	4942	8694	23
38	4326	9016	4483	8939	4638	8859	4792	8777	4944	8692	22
39	4329	9015	4485	8938	4641	8858	4795	8776	4947	8691	21
40	4331	9013	4488	8936	4643	8857	4797	8774	4950	8689	20
41	4334	9012	4491	8935	4646	8855	4800	8773	4952	8688	19
42	4337	9011	4493	8934	4648	8854	4802	8771	4955	8686	18
43	4339	9010	4496	8932	4651	8853	4805	8770	4957	8685	17
44	4342	9008	4498	8931	4654	8851	4807	8769	4960	8683	16
45	4344	9007	4501	8930	4656	8850	4810	8767	4962	8682	15
46	4347	9006	4504	8928	4659	8849	4812	8766	4965	8681	14
47	4350	9004	4506	8927	4661	8847	4815	8764	4967	8679	13
48	4352	9003	4509	8926	4664	8846	4818	8763	4970	8678	12
49	4355	9002	4511	8925	4666	8844	4820	8762	4972	8676	11
50	4358	9001	4514	8928	4669	8843	4823	8760	4975	8675	10
51	4360	8999	4517	8922	4672	8842	4825	8759	4977	8673	9
52	4363	8998	4519	8921	4674	8840	4828	8757	4980	8672	8
53	4365	8997	4522	8919	4677	8839	4830	8756	4982	8670	7
54	4368	8996	4524	8918	4679	8838	4833	8755	4985	8669	6
55	4371	8994	4527	8917	4682	8836	4835	8753	4987	8668	5
56	4373	8993	4530	8915	4684	8835	4838	8752	4990	8666	4
57	4376	8992	4532	8914	4687	8834	4840	8750	4992	8665	3
58	4378	8990	4535	8913	4690	8832	4843	8749	4995	8663	2
59	4381	8989	4537	8911	4692	8831	4846	8748	4997	8662	1
60	4384	8988	4540	8910	4695	8829	4848	8746	5000	8660	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	64°		63°		62°		61°		60°		′

′	30°		31°		32°		33°		34°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	5000	8660	5150	8572	5299	8480	5446	8387	5592	8290	60
1	5003	8659	5153	8570	5302	8479	5449	8385	5594	8289	59
2	5005	8657	5155	8569	5304	8477	5451	8384	5597	8287	58
3	5008	8656	5158	8567	5307	8476	5454	8382	5599	8285	57
4	5010	8654	5160	8566	5309	8474	5456	8380	5602	8284	56
5	5013	8653	5163	8564	5312	8473	5459	8379	5604	8282	55
6	5015	8652	5165	8563	5314	8471	5461	8377	5606	8281	54
7	5018	8650	5168	8561	5316	8470	5463	8376	5609	8279	53
8	5020	8649	5170	8560	5319	8468	5466	8374	5611	8277	52
9	5023	8647	5173	8558	5321	8467	5468	8372	5614	8276	51
10	5025	8646	5175	8557	5324	8465	5471	8371	5616	8274	50
11	5028	8644	5178	8555	5326	8463	5473	8369	5618	8272	49
12	5030	8643	5180	8554	5329	8462	5476	8368	5621	8271	48
13	5033	8641	5183	8552	5331	8460	5478	8366	5623	8269	47
14	5035	8640	5185	8551	5334	8459	5480	8364	5626	8268	46
15	5038	8638	5188	8549	5336	8457	5483	8363	5628	8266	45
16	5040	8637	5190	8548	5339	8456	5485	8361	5630	8264	44
17	5043	8635	5193	8546	5341	8454	5488	8360	5633	8263	43
18	5045	8634	5195	8545	5344	8453	5490	8358	5635	8261	42
19	5048	8632	5198	8543	5346	8451	5493	8356	5638	8259	41
20	5050	8631	5200	8542	5348	8450	5495	8355	5640	8258	40
21	5053	8630	5203	8540	5351	8448	5498	8353	5642	8256	39
22	5055	8628	5205	8539	5353	8446	5500	8352	5645	8254	38
23	5058	8627	5208	8537	5356	8445	5502	8350	5647	8253	37
24	5060	8625	5210	8536	5358	8443	5505	8348	5650	8251	36
25	5063	8624	5213	8534	5361	8442	5507	8347	5652	8249	35
26	5065	8622	5215	8532	5363	8440	5510	8345	5654	8248	34
27	5068	8621	5218	8531	5366	8439	5512	8344	5657	8246	33
28	5070	8619	5220	8529	5368	8437	5515	8342	5659	8245	32
29	5073	8618	5223	8528	5371	8435	5517	8340	5662	8243	31
30	5075	8616	5225	8526	5373	8434	5519	8339	5664	8241	30
31	5078	8615	5227	8525	5375	8432	5522	8337	5666	8240	29
32	5080	8613	5230	8523	5378	8431	5524	8336	5669	8238	28
33	5083	8612	5232	8522	5380	8429	5527	8334	5671	8236	27
34	5085	8610	5235	8520	5383	8428	5529	8332	5674	8235	26
35	5088	8609	5237	8519	5385	8426	5531	8331	5676	8233	25
36	5090	8607	5240	8517	5388	8425	5534	8329	5678	8231	24
37	5093	8606	5242	8516	5390	8423	5536	8328	5681	8230	23
38	5095	8604	5245	8514	5393	8421	5539	8326	5683	8228	22
39	5098	8603	5247	8513	5395	8420	5541	8324	5686	8226	21
40	5100	8601	5250	8511	5398	8418	5544	8323	5688	8225	20
41	5103	8600	5252	8510	5400	8417	5546	8321	5690	8223	19
42	5105	8599	5255	8508	5402	8415	5548	8320	5693	8221	18
43	5108	8597	5257	8507	5405	8414	5551	8318	5695	8220	17
44	5110	8596	5260	8505	5407	8412	5553	8316	5698	8218	16
45	5113	8594	5262	8504	5410	8410	5556	8315	5700	8216	15
46	5115	8593	5265	8502	5412	8409	5558	8313	5702	8215	14
47	5118	8591	5267	8500	5415	8407	5561	8311	5705	8213	13
48	5120	8590	5270	8499	5417	8406	5563	8310	5707	8211	12
49	5123	8588	5272	8497	5420	8404	5565	8308	5710	8210	11
50	5125	8587	5275	8496	5422	8403	5568	8307	5712	8208	10
51	5128	8585	5277	8494	5424	8401	5570	8305	5714	8207	9
52	5130	8584	5279	8493	5427	8399	5573	8303	5717	8205	8
53	5133	8582	5282	8491	5429	8398	5575	8302	5719	8203	7
54	5135	8581	5284	8490	5432	8396	5577	8300	5721	8202	6
55	5138	8579	5287	8488	5434	8395	5580	8299	5724	8200	5
56	5140	8578	5289	8487	5437	8393	5582	8297	5726	8198	4
57	5143	8576	5292	8485	5439	8391	5585	8295	5729	8197	3
58	5145	8575	5294	8484	5442	8390	5587	8294	5731	8195	2
59	5148	8573	5297	8482	5444	8388	5590	8292	5733	8193	1
60	5150	8572	5299	8480	5446	8387	5592	8290	5736	8192	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	59°		58°		57°		56°		55°		′

′	35° sin	35° cos	36° sin	36° cos	37° sin	37° cos	38° sin	38° cos	39° sin	39° cos	′
0	5736	8192	5878	8090	6018	7986	6157	7880	6293	7771	60
1	5738	8190	5880	8088	6020	7985	6159	7878	6295	7770	59
2	5741	8188	5883	8087	6023	7983	6161	7877	6298	7768	58
3	5743	8187	5885	8085	6025	7981	6163	7875	6300	7766	57
4	5745	8185	5887	8083	6027	7979	6166	7873	6302	7764	56
5	5748	8183	5890	8082	6030	7978	6168	7871	6305	7762	55
6	5750	8181	5892	8080	6032	7976	6170	7869	6307	7760	54
7	5752	8180	5894	8078	6034	7974	6173	7868	6309	7759	53
8	5755	8178	5897	8076	6037	7972	6175	7866	6311	7757	52
9	5757	8176	5899	8075	6039	7971	6177	7864	6314	7755	51
10	5760	8175	5901	8073	6041	7969	6180	7862	6316	7753	50
11	5762	8173	5904	8071	6044	7967	6182	7860	6318	7751	49
12	5764	8171	5906	8070	6046	7965	6184	7859	6320	7749	48
13	5767	8170	5908	8068	6048	7964	6186	7857	6323	7748	47
14	5769	8168	5911	8066	6051	7962	6189	7855	6325	7746	46
15	5771	8166	5913	8064	6053	7960	6191	7853	6327	7744	45
16	5774	8165	5915	8063	6055	7958	6193	7851	6329	7742	44
17	5776	8163	5918	8061	6058	7956	6196	7850	6332	7740	43
18	5779	8161	5920	8059	6060	7955	6198	7848	6334	7738	42
19	5781	8160	5922	8058	6062	7953	6200	7846	6336	7737	41
20	5783	8158	5925	8056	6065	7951	6202	7844	6338	7735	40
21	5786	8156	5927	8054	6067	7950	6205	7842	6341	7733	39
22	5788	8155	5930	8052	6069	7948	6207	7841	6343	7731	38
23	5790	8153	5932	8051	6071	7946	6209	7839	6345	7729	37
24	5793	8151	5934	8049	6074	7944	6211	7837	6347	7727	36
25	5795	8150	5937	8047	6076	7942	6214	7835	6350	7725	35
26	5798	8148	5939	8045	6078	7941	6216	7833	6352	7724	34
27	5800	8146	5941	8044	6081	7939	6218	7832	6354	7722	33
28	5802	8145	5944	8042	6083	7937	6221	7830	6356	7720	32
29	5805	8143	5946	8040	6085	7935	6223	7828	6359	7718	31
30	5807	8141	5948	8039	6088	7934	6225	7826	6361	7716	30
31	5809	8139	5951	8037	6090	7932	6227	7824	6363	7714	29
32	5812	8138	5953	8035	6092	7930	6230	7822	6365	7713	28
33	5814	8136	5955	8033	6095	7928	6232	7821	6368	7711	27
34	5816	8134	5958	8032	6097	7926	6234	7819	6370	7709	26
35	5819	8133	5960	8030	6099	7925	6237	7817	6372	7707	25
36	5821	8131	5962	8028	6101	7923	6239	7815	6374	7705	24
37	5824	8129	5965	8026	6104	7921	6241	7813	6376	7703	23
38	5826	8128	5967	8025	6106	7919	6243	7812	6379	7701	22
39	5828	8126	5969	8023	6108	7918	6246	7810	6381	7700	21
40	5831	8124	5972	8021	6111	7916	6248	7808	6383	7698	20
41	5833	8123	5974	8020	6113	7914	6250	7806	6385	7696	19
42	5835	8121	5976	8018	6115	7912	6252	7804	6388	7694	18
43	5838	8119	5979	8016	6118	7910	6255	7802	6390	7692	17
44	5840	8117	5981	8014	6120	7909	6257	7801	6392	7690	16
45	5842	8116	5983	8013	6122	7907	6259	7799	6394	7688	15
46	5845	8114	5986	8011	6124	7905	6262	7797	6397	7687	14
47	5847	8112	5988	8009	6127	7903	6264	7795	6399	7685	13
48	5850	8111	5990	8007	6129	7902	6266	7793	6401	7683	12
49	5852	8109	5993	8006	6131	7900	6268	7792	6403	7681	11
50	5854	8107	5995	8004	6134	7898	6271	7790	6406	7679	10
51	5857	8106	5997	8002	6136	7896	6273	7788	6408	7677	9
52	5859	8104	6000	8000	6138	7894	6275	7786	6410	7675	8
53	5861	8102	6002	7999	6141	7893	6277	7784	6412	7674	7
54	5864	8100	6004	7997	6143	7891	6280	7782	6414	7672	6
55	5866	8099	6007	7995	6145	7889	6282	7781	6417	7670	5
56	5868	8097	6009	7993	6147	7887	6284	7779	6419	7668	4
57	5871	8095	6011	7992	6150	7885	6286	7777	6421	7666	3
58	5873	8094	6014	7990	6152	7884	6289	7775	6423	7664	2
59	5875	8092	6016	7988	6154	7882	6291	7773	6426	7662	1
60	5878	8090	6018	7986	6157	7880	6293	7771	6428	7660	0
′	54° cos	54° sin	53° cos	53° sin	52° cos	52° sin	51° cos	51° sin	50° cos	50° sin	′

′	40°		41°		42°		43°		44°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	6428	7660	6561	7547	6691	7431	6820	7314	6947	7193	60
1	6430	7659	6563	7545	6693	7430	6822	7312	6949	7191	59
2	6432	7657	6565	7543	6696	7428	6824	7310	6951	7189	58
3	6435	7655	6567	7541	6698	7426	6826	7308	6953	7187	57
4	6437	7653	6569	7539	6700	7424	6828	7306	6955	7185	56
5	6439	7651	6572	7538	6702	7422	6831	7304	6957	7183	55
6	6441	7649	6574	7536	6704	7420	6833	7302	6959	7181	54
7	6443	7647	6576	7534	6706	7418	6835	7300	6961	7179	53
8	6446	7645	6578	7532	6709	7416	6837	7298	6963	7177	52
9	6448	7644	6580	7530	6711	7414	6839	7296	6965	7175	51
10	6450	7642	6583	7528	6713	7412	6841	7294	6967	7173	50
11	6452	7640	6585	7526	6715	7410	6843	7292	6970	7171	49
12	6455	7638	6587	7524	6717	7408	6845	7290	6972	7169	48
13	6457	7636	6589	7522	6719	7406	6848	7288	6974	7167	47
14	6459	7634	6591	7520	6722	7404	6850	7286	6976	7165	46
15	6461	7632	6593	7518	6724	7402	6852	7284	6978	7163	45
16	6463	7630	6596	7516	6726	7400	6854	7282	6980	7161	44
17	6466	7629	6598	7515	6728	7398	6856	7280	6982	7159	43
18	6468	7627	6600	7513	6730	7396	6858	7278	6984	7157	42
19	6470	7625	6602	7511	6732	7394	6860	7276	6986	7155	41
20	6472	7623	6604	7509	6734	7392	6862	7274	6988	7153	40
21	6475	7621	6607	7507	6737	7390	6865	7272	6990	7151	39
22	6477	7619	6609	7505	6739	7388	6867	7270	6992	7149	38
23	6479	7617	6611	7503	6741	7387	6869	7268	6995	7147	37
24	6481	7615	6613	7501	6743	7385	6871	7266	6997	7145	36
25	6483	7613	6615	7499	6745	7383	6873	7264	6999	7143	35
26	6486	7612	6617	7497	6747	7381	6875	7262	7001	7141	34
27	6488	7610	6620	7495	6749	7379	6877	7260	7003	7139	33
28	6490	7608	6622	7493	6752	7377	6879	7258	7005	7137	32
29	6492	7606	6624	7491	6754	7375	6881	7256	7007	7135	31
30	6494	7604	6626	7490	6756	7373	6884	7254	7009	7133	30
31	6497	7602	6628	7488	6758	7371	6886	7252	7011	7130	29
32	6499	7600	6631	7486	6760	7369	6888	7250	7013	7128	28
33	6501	7598	6633	7484	6762	7367	6890	7248	7015	7126	27
34	6503	7596	6635	7482	6764	7365	6892	7246	7017	7124	26
35	6506	7595	6637	7480	6767	7363	6894	7244	7019	7122	25
36	6508	7593	6639	7478	6769	7361	6896	7242	7022	7120	24
37	6510	7591	6641	7476	6771	7359	6898	7240	7024	7118	23
38	6512	7589	6644	7474	6773	7357	6900	7238	7026	7116	22
39	6514	7587	6646	7472	6775	7355	6903	7236	7028	7114	21
40	6517	7585	6648	7470	6777	7353	6905	7234	7030	7112	20
41	6519	7583	6650	7468	6779	7351	6907	7232	7032	7110	19
42	6521	7581	6652	7466	6782	7349	6909	7230	7034	7108	18
43	6523	7579	6654	7464	6784	7347	6911	7228	7036	7106	17
44	6525	7578	6657	7463	6786	7345	6913	7226	7038	7104	16
45	6528	7576	6659	7461	6788	7343	6915	7224	7040	7102	15
46	6530	7574	6661	7459	6790	7341	6917	7222	7042	7100	14
47	6532	7572	6663	7457	6792	7339	6919	7220	7044	7098	13
48	6534	7570	6665	7455	6794	7337	6921	7218	7046	7096	12
49	6536	7568	6667	7453	6797	7335	6924	7216	7048	7094	11
50	6539	7566	6670	7451	6799	7333	6926	7214	7050	7092	10
51	6541	7564	6672	7449	6801	7331	6928	7212	7053	7090	9
52	6543	7562	6674	7447	6803	7329	6930	7210	7055	7088	8
53	6545	7560	6676	7445	6805	7327	6932	7208	7057	7085	7
54	6547	7559	6678	7443	6807	7325	6934	7206	7059	7083	6
55	6550	7557	6680	7441	6809	7323	6936	7203	7061	7081	5
56	6552	7555	6683	7439	6811	7321	6938	7201	7063	7079	4
57	6554	7553	6685	7437	6814	7319	6940	7199	7065	7077	3
58	6556	7551	6687	7435	6816	7318	6942	7197	7067	7075	2
59	6558	7549	6689	7433	6818	7316	6944	7195	7069	7073	1
60	6561	7547	6691	7431	6820	7314	6947	7193	7071	7071	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	49°		48°		47°		46°		45°		′

′	0°		1°		2°		3°		4°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	0000	Infinite	0175	57.2900	0349	28.6363	0524	19.0811	0699	14.3007	60
1	0003	3437.75	0177	56.3506	0352	28.3994	0527	18.9755	0702	14.2411	59
2	0006	1718.87	0180	55.4415	0355	28.1664	0530	18.8711	0705	14.1821	58
3	0009	1145.92	0183	54.5613	0358	27.9372	0533	18.7678	0708	14.1235	57
4	0012	859.436	0186	53.7086	0361	27.7117	0536	18.6656	0711	14.0655	56
5	0015	687.549	0189	52.8821	0364	27.4899	0539	18.5645	0714	14.0079	55
6	0017	572.957	0192	52.0807	0367	27.2715	0542	18.4645	0717	13.9507	54
7	0020	491.106	0195	51.3032	0370	27.0566	0544	18.3655	0720	13.8940	53
8	0023	429.718	0198	50.5485	0373	26.8450	0547	18.2677	0723	13.8378	52
9	0026	381.971	0201	49.8157	0375	26.6367	0550	18.1708	0726	13.7821	51
10	0029	343.774	0204	49.1039	0378	26.4316	0553	18.0750	0729	13.7267	50
11	0032	312.521	0207	48.4121	0381	26.2296	0556	17.9802	0731	13.6719	49
12	0035	286.478	0209	47.7395	0384	26.0307	0559	17.8863	0734	13.6174	48
13	0038	264.441	0212	47.0853	0387	25.8348	0562	17.7934	0737	13.5634	47
14	0041	245.552	0215	46.4489	0390	25.6418	0565	17.7015	0740	13.5098	46
15	0044	229.182	0218	45.8294	0393	25.4517	0568	17.6106	0743	13.4566	45
16	0047	214.858	0221	45.2261	0396	25.2644	0571	17.5205	0746	13.4039	44
17	0049	202.219	0224	44.6386	0399	25.0798	0574	17.4314	0749	13.3515	43
18	0052	190.984	0227	44.0661	0402	24.8978	0577	17.3432	0752	13.2996	42
19	0055	180.932	0230	43.5081	0405	24.7185	0580	17.2558	0755	13.2480	41
20	0058	171.885	0233	42.9641	0407	24.5418	0582	17.1693	0758	13.1969	40
21	0061	163.700	0236	42.4335	0410	24.3675	0585	17.0837	0761	13.1461	39
22	0064	156.259	0239	41.9158	0413	24.1957	0588	16.9990	0764	13.0958	38
23	0067	149.465	0241	41.4106	0416	24.0263	0591	16.9150	0767	13.0458	37
24	0070	143.237	0244	40.9174	0419	23.8593	0594	16.8319	0769	12.9962	36
25	0073	137.507	0247	40.4358	0422	23.6945	0597	16.7496	0772	12.9469	35
26	0076	132.219	0250	39.9655	0425	23.5321	0600	16.6681	0775	12.8981	34
27	0079	127.321	0253	39.5059	0428	23.3718	0603	16.5874	0778	12.8496	33
28	0081	122.774	0256	39.0568	0431	23.2137	0606	16.5075	0781	12.8014	32
29	0084	118.540	0259	38.6177	0434	23.0577	0609	16.4283	0784	12.7536	31
30	0087	114.589	0262	38.1885	0437	22.9038	0612	16.3499	0787	12.7062	30
31	0090	110.892	0265	37.7686	0440	22.7519	0615	16.2722	0790	12.6591	29
32	0093	107.426	0268	37.3579	0442	22.6020	0617	16.1952	0793	12.6124	28
33	0096	104.171	0271	36.9560	0445	22.4541	0620	16.1190	0796	12.5660	27
34	0099	101.107	0274	36.5627	0448	22.3081	0623	16.0435	0799	12.5199	26
35	0102	98.2179	0276	36.1776	0451	22.1640	0626	15.9687	0802	12.4742	25
36	0105	95.4895	0279	35.8006	0454	22.0217	0629	15.8945	0805	12.4288	24
37	0108	92.9085	0282	35.4313	0457	21.8813	0632	15.8211	0808	12.3838	23
38	0111	90.4633	0285	35.0695	0460	21.7426	0635	15.7483	0810	12.3390	22
39	0113	88.1436	0288	34.7151	0463	21.6056	0638	15.6762	0813	12.2946	21
40	0116	85.9398	0291	34.3678	0466	21.4704	0641	15.6048	0816	12.2505	20
41	0119	83.8435	0294	34.0273	0469	21.3369	0644	15.5340	0819	12.2067	19
42	0122	81.8470	0297	33.6935	0472	21.2049	0647	15.4638	0822	12.1632	18
43	0125	79.9434	0300	33.3662	0475	21.0747	0650	15.3943	0825	12.1201	17
44	0128	78.1263	0303	33.0452	0477	20.9460	0653	15.3254	0828	12.0772	16
45	0131	76.3900	0306	32.7303	0480	20.8188	0655	15.2571	0831	12.0346	15
46	0134	74.7292	0308	32.4213	0483	20.6932	0658	15.1893	0834	11.9923	14
47	0137	73.1390	0311	32.1181	0486	20.5691	0661	15.1222	0837	11.9504	13
48	0140	71.6151	0314	31.8205	0489	20.4465	0664	15.0557	0840	11.9087	12
49	0143	70.1533	0317	31.5284	0492	20.3253	0667	14.9898	0843	11.8673	11
50	0146	68.7501	0320	31.2416	0495	20.2056	0670	14.9244	0846	11.8262	10
51	0148	67.4019	0323	30.9599	0498	20.0872	0673	14.8596	0849	11.7853	9
52	0151	66.1055	0326	30.6833	0501	19.9702	0676	14.7954	0851	11.7448	8
53	0154	64.8580	0329	30.4116	0504	19.8546	0679	14.7317	0854	11.7045	7
54	0157	63.6567	0332	30.1446	0507	19.7403	0682	14.6685	0857	11.6645	6
55	0160	62.4992	0335	29.8823	0509	19.6273	0685	14.6059	0860	11.6248	5
56	0163	61.3829	0338	29.6245	0512	19.5156	0688	14.5438	0863	11.5853	4
57	0166	60.3058	0340	29.3711	0515	19.4051	0690	14.4823	0866	11.5461	3
58	0169	59.2659	0343	29.1220	0518	19.2959	0693	14.4212	0869	11.5072	2
59	0172	58.2612	0346	28.8771	0521	19.1879	0696	14.3607	0872	11.4685	1
60	0175	57.2900	0349	28.6363	0524	19.0811	0699	14.3007	0875	11.4301	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	89°		88°		87°		86°		85°		′

′	5° tan	5° cot	6° tan	6° cot	7° tan	7° cot	8° tan	8° cot	9° tan	9° cot	′
0	0875	11.4301	1051	9.5144	1228	8.1443	1405	7.1154	1584	6.3138	**60**
1	0878	11.3919	1054	9.4878	1231	8.1248	1408	7.1004	1587	6.3019	59
2	0881	11.3540	1057	9.4614	1234	8.1054	1411	7.0855	1590	6.2901	58
3	0884	11.3163	1060	9.4352	1237	8.0860	1414	7.0706	1593	6.2783	57
4	0887	11.2789	1063	9.4090	1240	8.0667	1417	7.0558	1596	6.2666	56
5	0890	11.2417	1066	9.3831	1243	8.0476	1420	7.0410	1599	6.2549	**55**
6	0892	11.2048	1069	9.3572	1246	8.0285	1423	7.0264	1602	6.2432	54
7	0895	11.1681	1072	9.3315	1249	8.0095	1426	7.0117	1605	6.2316	53
8	0898	11.1316	1075	9.3060	1251	7.9906	1429	6.9972	1608	6.2200	52
9	0901	11.0954	1078	9.2806	1254	7.9718	1432	6.9827	1611	6.2085	51
10	0904	11.0594	1080	9.2553	1257	7.9530	1435	6.9682	1614	6.1970	**50**
11	0907	11.0237	1083	9.2302	1260	7.9344	1438	6.9538	1617	6.1856	49
12	0910	10.9882	1086	9.2052	1263	7.9158	1441	6.9395	1620	6.1742	48
13	0913	10.9529	1089	9.1803	1266	7.8973	1444	6.9252	1623	6.1628	47
14	0916	10.9178	1092	9.1555	1269	7.8789	1447	6.9110	1626	6.1515	46
15	0919	10.8829	1095	9.1309	1272	7.8606	1450	6.8969	1629	6.1402	**45**
16	0922	10.8483	1098	9.1065	1275	7.8424	1453	6.8828	1632	6.1290	44
17	0925	10.8139	1101	9.0821	1278	7.8243	1456	6.8687	1635	6.1178	43
18	0928	10.7797	1104	9.0579	1281	7.8062	1459	6.8548	1638	6.1066	42
19	0931	10.7457	1107	9.0338	1284	7.7883	1462	6.8408	1641	6.0955	41
20	0934	10.7119	1110	9.0098	1287	7.7704	1465	6.8269	1644	6.0844	**40**
21	0936	10.6783	1113	8.9860	1290	7.7525	1468	6.8131	1647	6.0734	39
22	0939	10.6450	1116	8.9623	1293	7.7348	1471	6.7994	1650	6.0624	38
23	0942	10.6118	1119	8.9387	1296	7.7171	1474	6.7856	1653	6.0514	37
24	0945	10.5789	1122	8.9152	1299	7.6996	1477	6.7720	1655	6.0405	36
25	0948	10.5462	1125	8.8919	1302	7.6821	1480	6.7584	1658	6.0296	**35**
26	0951	10.5136	1128	8.8686	1305	7.6647	1483	6.7448	1661	6.0188	34
27	0954	10.4813	1131	8.8455	1308	7.6473	1486	6.7313	1664	6.0080	33
28	0957	10.4491	1134	8.8225	1311	7.6301	1489	6.7179	1667	5.9972	32
29	0960	10.4172	1136	8.7996	1314	7.6129	1492	6.7045	1670	5.9865	31
30	0963	10.3854	1139	8.7769	1317	7.5958	1495	6.6912	1673	5.9758	**30**
31	0966	10.3538	1142	8.7542	1319	7.5787	1497	6.6779	1676	5.9651	29
32	0969	10.3224	1145	8.7317	1322	7.5618	1500	6.6646	1679	5.9545	28
33	0972	10.2913	1148	8.7093	1325	7.5449	1503	6.6514	1682	5.9439	27
34	0975	10.2602	1151	8.6870	1328	7.5281	1506	6.6383	1685	5.9333	26
35	0978	10.2294	1154	8.6648	1331	7.5113	1509	6.6252	1688	5.9228	**25**
36	0981	10.1988	1157	8.6427	1334	7.4947	1512	6.6122	1691	5.9124	24
37	0983	10.1683	1160	8.6208	1337	7.4781	1515	6.5992	1694	5.9019	23
38	0986	10.1381	1163	8.5989	1340	7.4615	1518	6.5863	1697	5.8915	22
39	0989	10.1080	1166	8.5772	1343	7.4451	1521	6.5734	1700	5.8811	21
40	0992	10.0780	1169	8.5555	1346	7.4287	1524	6.5606	1703	5.8708	**20**
41	0995	10.0483	1172	8.5340	1349	7.4124	1527	6.5478	1706	5.8605	19
42	0998	10.0187	1175	8.5126	1352	7.3962	1530	6.5350	1709	5.8502	18
43	1001	9.9893	1178	8.4913	1355	7.3800	1533	6.5223	1712	5.8400	17
44	1004	9.9601	1181	8.4701	1358	7.3639	1536	6.5097	1715	5.8298	16
45	1007	9.9310	1184	8.4490	1361	7.3479	1539	6.4971	1718	5.8197	**15**
46	1010	9.9021	1187	8.4280	1364	7.3319	1542	6.4846	1721	5.8095	14
47	1013	9.8734	1189	8.4071	1367	7.3160	1545	6.4721	1724	5.7994	13
48	1016	9.8448	1192	8.3863	1370	7.3002	1548	6.4596	1727	5.7894	12
49	1019	9.8164	1195	8.3656	1373	7.2844	1551	6.4472	1730	5.7794	12
50	1022	9.7882	1198	8.3450	1376	7.2687	1554	6.4348	1733	5.7694	**10**
51	1025	9.7601	1201	8.3245	1379	7.2531	1557	6.4225	1736	5.7594	9
52	1028	9.7322	1204	8.3041	1382	7.2375	1560	6.4103	1739	5.7495	8
53	1030	9.7044	1207	8.2838	1385	7.2220	1563	6.3980	1742	5.7396	7
54	1033	9.6768	1210	8.2636	1388	7.2066	1566	6.3859	1745	5.7297	6
55	1036	9.6499	1213	8.2434	1391	7.1912	1569	6.3737	1748	5.7199	**5**
56	1039	9.6220	1216	8.2234	1394	7.1759	1572	6.3617	1751	5.7101	4
57	1042	9.5949	1219	8.2035	1397	7.1607	1575	6.3496	1754	5.7004	3
58	1045	9.5679	1222	8.1837	1399	7.1455	1578	6.3376	1757	5.6906	2
59	1048	9.5411	1225	8.1640	1402	7.1304	1581	6.3257	1760	5.6809	1
60	1051	9.5144	1228	8.1443	1405	7.1154	1584	6.3138	1763	5.6713	**0**
′	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	′
	84°		83°		82°		81°		80°		

′	10°		11°		12°		13°		14°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	1763	5.6713	1944	5.1446	2126	4.7046	2309	4.3315	2493	4.0108	**60**
1	1766	5.6617	1947	5.1366	2129	4.6979	2312	4.3257	2496	4.0058	59
2	1769	5.6521	1950	5.1286	2132	4.6912	2315	4.3200	2499	4.0009	58
3	1772	5.6425	1953	5.1207	2135	4.6845	2318	4.3143	2503	3.9959	57
4	1775	5.6330	1956	5.1128	2138	4.6779	2321	4.3086	2506	3.9910	56
5	1778	5.6234	1959	5.1049	2141	4.6712	2324	4.3029	2509	3.9861	**55**
6	1781	5.6140	1962	5.0970	2144	4.6646	2327	4.2972	2512	3.9812	54
7	1784	5.6045	1965	5.0892	2147	4.6580	2330	4.2916	2515	3.9763	53
8	1787	5.5951	1968	5.0814	2150	4.6514	2333	4.2859	2518	3.9714	52
9	1790	5.5857	1971	5.0736	2153	4.6448	2336	4.2803	2521	3.9665	51
10	1793	5.5764	1974	5.0658	2156	4.6382	2339	4.2747	2524	3.9617	**50**
11	1796	5.5671	1977	5.0581	2159	4.6317	2342	4.2691	2527	3.9568	49
12	1799	5.5578	1980	5.0504	2162	4.6252	2345	4.2635	2530	3.9520	48
13	1802	5.5485	1983	5.0427	2165	4.6187	2349	4.2580	2533	3.9471	47
14	1805	5.5393	1986	5.0350	2168	4.6122	2352	4.2524	2537	3.9423	46
15	1808	5.5301	1989	5.0273	2171	4.6057	2355	4.2468	2540	3.9375	**45**
16	1811	5.5209	1992	5.0197	2174	4.5993	2358	4.2413	2543	3.9327	44
17	1814	5.5118	1995	5.0121	2177	4.5928	2361	4.2358	2546	3.9279	43
18	1817	5.5026	1998	5.0045	2180	4.5864	2364	4.2303	2549	3.9232	42
19	1820	5.4936	2001	4.9969	2183	4.5800	2367	4.2248	2552	3.9184	41
20	1823	5.4845	2004	4.9894	2186	4.5736	2370	4.2193	2555	3.9136	**40**
21	1826	5.4755	2007	4.9819	2189	4.5673	2373	4.2139	2558	3.9089	39
22	1829	5.4665	2010	4.9744	2193	4.5609	2376	4.2084	2561	3.9042	38
23	1832	5.4575	2013	4.9669	2196	4.5546	2379	4.2030	2564	3.8995	37
24	1835	5.4486	2016	4.9594	2199	4.5483	2382	4.1976	2568	3.8947	36
25	1838	5.4397	2019	4.9520	2202	4.5420	2385	4.1922	2571	3.8900	**35**
26	1841	5.4308	2022	4.9446	2205	4.5357	2388	4.1868	2574	3.8854	34
27	1844	5.4219	2025	4.9372	2208	4.5294	2392	4.1814	2577	3.8807	33
28	1847	5.4131	2028	4.9298	2211	4.5232	2395	4.1760	2580	3.8760	32
29	1850	5.4043	2031	4.9225	2214	4.5169	2398	4.1706	2583	3.8714	31
30	1853	5.3955	2035	4.9152	2217	4.5107	2401	4.1653	2586	3.8667	**30**
31	1856	5.3868	2038	4.9078	2220	4.5045	2404	4.1600	2589	3.8621	29
32	1859	5.3781	2041	4.9006	2223	4.4983	2407	4.1547	2592	3.8575	28
33	1862	5.3694	2044	4.8933	2226	4.4922	2410	4.1493	2595	3.8528	27
34	1865	5.3607	2047	4.8860	2229	4.4860	2413	4.1441	2599	3.8482	26
35	1868	5.3521	2050	4.8788	2232	4.4799	2416	4.1388	2602	3.8436	**25**
36	1871	5.3435	2053	4.8716	2235	4.4737	2419	4.1335	2605	3.8391	24
37	1874	5.3349	2056	4.8644	2238	4.4676	2422	4.1282	2608	3.8345	23
38	1877	5.3263	2059	4.8573	2241	4.4615	2425	4.1230	2611	3.8299	22
39	1880	5.3178	2062	4.8501	2244	4.4555	2428	4.1178	2614	3.8254	21
40	1883	5.3093	2065	4.8430	2247	4.4494	2432	4.1126	2617	3.8208	**20**
41	1887	5.3008	2068	4.8359	2251	4.4434	2435	4.1074	2620	3.8163	19
42	1890	5.2924	2071	4.8288	2254	4.4374	2438	4.1022	2623	3.8118	18
43	1893	5.2839	2074	4.8218	2257	4.4313	2441	4.0970	2627	3.8073	17
44	1896	5.2755	2077	4.8147	2260	4.4253	2444	4.0918	2630	3.8028	16
45	1899	5.2672	2080	4.8077	2263	4.4194	2447	4.0867	2633	3.7983	**15**
46	1902	5.2588	2083	4.8007	2266	4.4134	2450	4.0815	2636	3.7938	14
47	1905	5.2505	2086	4.7937	2269	4.4075	2453	4.0764	2639	3.7893	13
48	1908	5.2422	2089	4.7867	2272	4.4015	2456	4.0713	2642	3.7848	12
49	1911	5.2339	2092	4.7798	2275	4.3956	2459	4.0662	2645	3.7804	11
50	1914	5.2257	2095	4.7729	2278	4.3897	2462	4.0611	2648	3.7760	**10**
51	1917	5.2174	2098	4.7659	2281	4.3838	2465	4.0560	2651	3.7715	9
52	1920	5.2092	2101	4.7591	2284	4.3779	2469	4.0509	2655	3.7671	8
53	1923	5.2011	2104	4.7522	2287	4.3721	2472	4.0459	2658	3.7627	7
54	1926	5.1929	2107	4.7453	2290	4.3662	2475	4.0408	2661	3.7583	6
55	1929	5.1848	2110	4.7385	2293	4.3604	2478	4.0358	2664	3.7539	**5**
56	1932	5.1767	2113	4.7317	2296	4.3546	2481	4.0308	2667	3.7495	4
57	1935	5.1686	2116	4.7249	2299	4.3488	2484	4.0257	2670	3.7451	3
58	1938	5.1606	2119	4.7181	2303	4.3430	2487	4.0207	2673	3.7408	2
59	1941	5.1526	2123	4.7114	2306	4.3372	2490	4.0158	2676	3.7364	1
60	1944	5.1446	2126	4.7046	2309	4.3315	2493	4.0108	2679	3.7321	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	79°		78°		77°		76°		75°		′

′	15° tan	15° cot	16° tan	16° cot	17° tan	17° cot	18° tan	18° cot	19° tan	19° cot	′
0	2679	3.7321	2867	3.4874	3057	3.2709	3249	3.0777	3443	2.9042	60
1	2683	3.7277	2871	3.4836	3060	3.2675	3252	3.0746	3447	2.9015	59
2	2686	3.7234	2874	3.4798	3064	3.2641	3256	3.0716	3450	2.8987	58
3	2689	3.7191	2877	3.4760	3067	3.2607	3259	3.0686	3453	2.8960	57
4	2692	3.7148	2880	3.4722	3070	3.2573	3262	3.0655	3456	2.8933	56
5	2695	3.7105	2883	3.4684	3073	3.2539	3265	3.0625	3460	2.8905	55
6	2698	3.7062	2886	3.4646	3076	3.2506	3269	3.0595	3463	2.8878	54
7	2701	3.7019	2890	3.4608	3080	3.2472	3272	3.0565	3466	2.8851	53
8	2704	3.6976	2893	3.4570	3083	3.2438	3275	3.0535	3469	2.8824	52
9	2708	3.6933	2896	3.4533	3086	3.2405	3278	3.0505	3473	2.8797	51
10	2711	3.6891	2899	3.4495	3089	3.2371	3281	3.0475	3476	2.8770	50
11	2714	3.6848	2902	3.4458	3092	3.2338	3285	3.0445	3479	2.8743	49
12	2717	3.6806	2905	3.4420	3096	3.2305	3288	3.0415	3482	2.8716	48
13	2720	3.6764	2908	3.4383	3099	3.2272	3291	3.0385	3486	2.8689	47
14	2723	3.6722	2912	3.4346	3102	3.2238	3294	3.0356	3489	2.8662	46
15	2726	3.6680	2915	3.4308	3105	3.2205	3298	3.0326	3492	2.8636	45
16	2729	3.6638	2918	3.4271	3108	3.2172	3301	3.0296	3495	2.8609	44
17	2733	3.6596	2921	3.4234	3111	3.2139	3304	3.0267	3499	2.8582	43
18	2736	3.6554	2924	3.4197	3115	3.2106	3307	3.0237	3502	2.8556	42
19	2739	3.6512	2927	3.4160	3118	3.2073	3310	3.0208	3505	2.8529	41
20	2742	3.6470	2931	3.4124	3121	3.2041	3314	3.0178	3508	2.8502	40
21	2745	3.6429	2934	3.4087	3124	3.2008	3317	3.0149	3512	2.8476	39
22	2748	3.6387	2937	3.4050	3127	3.1975	3320	3.0120	3515	2.8449	38
23	2751	3.6346	2940	3.4014	3131	3.1943	3323	3.0090	3518	2.8423	37
24	2754	3.6305	2943	3.3977	3134	3.1910	3327	3.0061	3522	2.8397	36
25	2758	3.6264	2946	3.3941	3137	3.1878	3330	3.0032	3525	2.8370	35
26	2761	3.6222	2949	3.3904	3140	3.1845	3333	3.0003	3528	2.8344	34
27	2764	3.6181	2953	3.3868	3143	3.1813	3336	2.9974	3531	2.8318	33
28	2767	3.6140	2956	3.3832	3147	3.1780	3339	2.9945	3535	2.8291	32
29	2770	3.6100	2959	3.3796	3150	3.1748	3343	2.9916	3538	2.8265	31
30	2773	3.6059	2962	3.3759	3153	3.1716	3346	2.9887	3541	2.8239	30
31	2776	3.6018	2965	3.3723	3156	3.1684	3349	2.9858	3544	2.8213	29
32	2780	3.5978	2968	3.3687	3159	3.1652	3352	2.9829	3548	2.8187	28
33	2783	3.5937	2972	3.3652	3163	3.1620	3356	2.9800	3551	2.8161	27
34	2786	3.5897	2975	3.3616	3166	3.1588	3359	2.9772	3554	2.8135	26
35	2789	3.5856	2978	3.3580	3169	3.1556	3362	2.9743	3558	2.8109	25
36	2792	3.5816	2981	3.3544	3172	3.1524	3365	2.9714	3561	2.8083	24
37	2795	3.5776	2984	3.3509	3175	3.1492	3369	2.9686	3564	2.8057	23
38	2798	3.5736	2987	3.3473	3179	3.1460	3372	2.9657	3567	2.8032	22
39	2801	3.5696	2991	3.3438	3182	3.1429	3375	2.9629	3571	2.8006	21
40	2805	3.5656	2994	3.3402	3185	3.1397	3378	2.9600	3574	2.7980	20
41	2808	3.5616	2997	3.3367	3188	3.1366	3382	2.9572	3577	2.7955	19
42	2811	3.5576	3000	3.3332	3191	3.1334	3385	2.9544	3581	2.7929	18
43	2814	3.5536	3003	3.3297	3195	3.1303	3388	2.9515	3584	2.7903	17
44	2817	3.5497	3006	3.3261	3198	3.1271	3391	2.9487	3587	2.7878	16
45	2820	3.5457	3010	3.3226	3201	3.1240	3395	2.9459	3590	2.7852	15
46	2823	3.5418	3013	3.3191	3204	3.1209	3398	2.9431	3594	2.7827	14
47	2827	3.5379	3016	3.3156	3207	3.1178	3401	2.9403	3597	2.7801	13
48	2830	3.5339	3019	3.3122	3211	3.1146	3404	2.9375	3600	2.7776	12
49	2833	3.5300	3022	3.3087	3214	3.1115	3408	2.9347	3604	2.7751	11
50	2836	3.5261	3026	3.3052	3217	3.1084	3411	2.9319	3607	2.7725	10
51	2839	3.5222	3029	3.3017	3220	3.1053	3414	2.9291	3610	2.7700	9
52	2842	3.5183	3032	3.2983	3223	3.1022	3417	2.9263	3613	2.7675	8
53	2845	3.5144	3035	3.2948	3227	3.0991	3421	2.9235	3617	2.7650	7
54	2849	3.5105	3038	3.2914	3230	3.0961	3424	2.9208	3620	2.7625	6
55	2852	3.5067	3041	3.2880	3233	3.0930	3427	2.9180	3623	2.7600	5
56	2855	3.5028	3045	3.2845	3236	3.0899	3430	2.9152	3627	2.7575	4
57	2858	3.4989	3048	3.2811	3240	3.0868	3434	2.9125	3630	2.7550	3
58	2861	3.4951	3051	3.2777	3243	3.0838	3437	2.9097	3633	2.7525	2
59	2864	3.4912	3054	3.2743	3246	3.0807	3440	2.9070	3636	2.7500	1
60	2867	3.4874	3057	3.2709	3249	3.0777	3443	2.9042	3640	2.7475	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	74°		73°		72°		71°		70°		′

′	20°		21°		22°		23°		24°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	3640	2.7475	3839	2.6051	4040	2.4751	4245	2.3559	4452	2.2460	60
1	3643	2.7450	3842	2.6028	4044	2.4730	4248	2.3539	4456	2.2443	59
2	3646	2.7425	3845	2.6006	4047	2.4709	4252	2.3520	4459	2.2425	58
3	3650	2.7400	3849	2.5983	4050	2.4689	4255	2.3501	4463	2.2408	57
4	3653	2.7376	3852	2.5961	4054	2.4668	4258	2.3483	4466	2.2390	56
5	3656	2.7351	3855	2.5938	4057	2.4648	4262	2.3464	4470	2.2373	55
6	3659	2.7326	3859	2.5916	4061	2.4627	4265	2.3445	4473	2.2355	54
7	3663	2.7302	3862	2.5893	4064	2.4606	4269	2.3426	4477	2.2338	53
8	3666	2.7277	3865	2.5871	4067	2.4586	4272	2.3407	4480	2.2320	52
9	3669	2.7253	3869	2.5848	4071	2.4566	4276	2.3388	4484	2.2303	51
10	3673	2.7228	3872	2.5826	4074	2.4545	4279	2.3369	4487	2.2286	50
11	3676	2.7204	3875	2.5804	4078	2.4525	4283	2.3351	4491	2.2268	49
12	3679	2.7179	3879	2.5782	4081	2.4504	4286	2.3332	4494	2.2251	48
13	3683	2.7155	3882	2.5759	4084	2.4484	4289	2.3313	4498	2.2234	47
14	3686	2.7130	3885	2.5737	4088	2.4464	4293	2.3294	4501	2.2216	46
15	3689	2.7106	3889	2.5715	4091	2.4443	4296	2.3276	4505	2.2199	45
16	3693	2.7082	3892	2.5693	4095	2.4423	4300	2.3257	4508	2.2182	44
17	3696	2.7058	3895	2.5671	4098	2.4403	4303	2.3238	4512	2.2165	43
18	3699	2.7034	3899	2.5649	4101	2.4383	4307	2.3220	4515	2.2148	42
19	3702	2.7009	3902	2.5627	4105	2.4362	4310	2.3201	4519	2.2130	41
20	3706	2.6985	3906	2.5605	4108	2.4342	4314	2.3183	4522	2.2113	40
21	3709	2.6961	3909	2.5533	4111	2.4322	4317	2.3164	4526	2.2096	39
22	3712	2.6937	3912	2.5561	4115	2.4302	4320	2.3146	4529	2.2079	38
23	3716	2.6913	3916	2.5539	4118	2.4282	4324	2.3127	4533	2.2062	37
24	3719	2.6889	3919	2.5517	4122	2.4262	4327	2.3109	4536	2.2045	36
25	3722	2.6865	3922	2.5495	4125	2.4242	4331	2.3090	4540	2.2028	35
26	3726	2.6841	3926	2.5473	4129	2.4222	4334	2.3072	4543	2.2011	34
27	3729	2.6818	3929	2.5452	4132	2.4202	4338	2.3053	4547	2.1994	33
28	3732	2.6794	3932	2.5430	4135	2.4182	4341	2.3035	4550	2.1977	32
29	3736	2.6770	3936	2.5408	4139	2.4162	4345	2.3017	4554	2.1960	31
30	3739	2.6746	3939	2.5386	4142	2.4142	4348	2.2998	4557	2.1943	30
31	3742	2.6723	3942	2.5365	4146	2.4122	4352	2.2980	4561	2.1926	29
32	3745	2.6699	3946	2.5343	4149	2.4102	4355	2.2962	4564	2.1909	28
33	3749	2.6675	3949	2.5322	4152	2.4083	4359	2.2944	4568	2.1892	27
34	3752	2.6652	3953	2.5300	4156	2.4063	4362	2.2925	4571	2.1876	26
35	3755	2.6628	3956	2.5279	4159	2.4043	4365	2.2907	4575	2.1859	25
36	3759	2.6605	3959	2.5257	4163	2.4023	4369	2.2889	4578	2.1842	24
37	3762	2.6581	3963	2.5236	4166	2.4004	4372	2.2871	4582	2.1825	23
38	3765	2.6558	3966	2.5214	4169	2.3984	4376	2.2853	4585	2.1808	22
39	3769	2.6534	3969	2.5193	4173	2.3964	4379	2.2835	4589	2.1792	21
40	3772	2.6511	3973	2.5172	4176	2.3945	4383	2.2817	4592	2.1775	20
41	3775	2.6488	3976	2.5150	4180	2.3925	4386	2.2799	4596	2.1758	19
42	3779	2.6464	3979	2.5129	4183	2.3906	4390	2.2781	4599	2.1742	18
43	3782	2.6441	3983	2.5108	4187	2.3886	4393	2.2763	4603	2.1725	17
44	3785	2.6418	3986	2.5086	4190	2.3867	4397	2.2745	4607	2.1708	16
45	3789	2.6395	3990	2.5065	4193	2.3847	4400	2.2727	4610	2.1692	15
46	3792	2.6371	3993	2.5044	4197	2.3828	4404	2.2709	4614	2.1675	14
47	3795	2.6348	3996	2.5023	4200	2.3808	4407	2.2691	4617	2.1659	13
48	3799	2.6325	4000	2.5002	4204	2.3789	4411	2.2673	4621	2.1642	12
49	3802	2.6302	4003	2.4981	4207	2.3770	4414	2.2655	4624	2.1625	11
50	3805	2.6279	4006	2.4960	4210	2.3750	4417	2.2637	4628	2.1609	10
51	3809	2.6256	4010	2.4939	4214	2.3731	4421	2.2620	4631	2.1592	9
52	3812	2.6233	4013	2.4918	4217	2.3712	4424	2.2602	4635	2.1576	8
53	3815	2.6210	4017	2.4897	4221	2.3693	4428	2.2584	4638	2.1560	7
54	3819	2.6187	4020	2.4876	4224	2.3673	4431	2.2566	4642	2.1543	6
55	3822	2.6165	4023	2.4855	4228	2.3654	4435	2.2549	4645	2.1527	5
56	3825	2.6142	4027	2.4834	4231	2.3635	4438	2.2531	4649	2.1510	4
57	3829	2.6119	4030	2.4813	4234	2.3616	4442	2.2513	4652	2.1494	3
58	3832	2.6096	4033	2.4792	4238	2.3597	4445	2.2496	4656	2.1478	2
59	3835	2.6074	4037	2.4772	4241	2.3578	4449	2.2478	4660	2.1461	1
60	3839	2.6051	4040	2.4751	4245	2.3559	4452	2.2460	4663	2.1445	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	69°		68°		67°		66°		65°		′

′	25°		26°		27°		28°		29°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	4663	2.1445	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	**60**
1	4667	2.1429	4881	2.0488	5099	1.9612	5321	1.8794	5547	1.8028	59
2	4670	2.1413	4885	2.0473	5103	1.9598	5325	1.8781	5551	1.8016	58
3	4674	2.1396	4888	2.0458	5106	1.9584	5328	1.8768	5555	1.8003	57
4	4677	2.1380	4892	2.0443	5110	1.9570	5332	1.8755	5558	1.7991	56
5	4681	2.1364	4895	2.0428	5114	1.9556	5336	1.8741	5562	1.7979	**55**
6	4684	2.1348	4899	2.0413	5117	1.9542	5340	1.8728	5566	1.7966	54
7	4688	2.1332	4903	2.0398	5121	1.9528	5343	1.8715	5570	1.7954	53
8	4691	2.1315	4906	2.0383	5125	1.9514	5347	1.8702	5574	1.7942	52
9	4695	2.1299	4910	2.0368	5128	1.9500	5351	1.8689	5577	1.7930	51
10	4699	2.1283	4913	2.0353	5132	1.9486	5354	1.8676	5581	1.7917	**50**
11	4702	2.1267	4917	2.0338	5136	1.9472	5358	1.8663	5585	1.7905	49
12	4706	2.1251	4921	2.0323	5139	1.9458	5362	1.8650	5589	1.7893	48
13	4709	2.1235	4924	2.0308	5143	1.9444	5366	1.8637	5593	1.7881	47
14	4713	2.1219	4928	2.0293	5147	1.9430	5369	1.8624	5596	1.7868	46
15	4716	2.1203	4931	2.0278	5150	1.9416	5373	1.8611	5600	1.7856	**45**
16	4720	2.1187	4935	2.0263	5154	1.9402	5377	1.8598	5604	1.7844	44
17	4723	2.1171	4939	2.0248	5158	1.9388	5381	1.8585	5608	1.7832	43
18	4727	2.1155	4942	2.0233	5161	1.9375	5384	1.8572	5612	1.7820	42
19	4731	2.1139	4946	2.0219	5165	1.9361	5388	1.8559	5616	1.7808	41
20	4734	2.1123	4950	2.0204	5169	1.9347	5392	1.8546	5619	1.7796	**40**
21	4738	2.1107	4953	2.0189	5172	1.9333	5396	1.8533	5623	1.7783	39
22	4741	2.1092	4957	2.0174	5176	1.9319	5399	1.8520	5627	1.7771	38
23	4745	2.1076	4960	2.0160	5180	1.9306	5403	1.8507	5631	1.7759	37
24	4748	2.1060	4964	2.0145	5184	1.9292	5407	1.8495	5635	1.7747	36
25	4752	2.1044	4968	2.0130	5187	1.9278	5411	1.8482	5639	1.7735	**35**
26	4755	2.1028	4971	2.0115	5191	1.9265	5415	1.8469	5642	1.7723	34
27	4759	2.1013	4975	2.0101	5195	1.9251	5418	1.8456	5646	1.7711	33
28	4763	2.0997	4979	2.0086	5198	1.9237	5422	1.8443	5650	1.7699	32
29	4766	2.0981	4982	2.0072	5202	1.9223	5426	1.8430	5654	1.7687	31
30	4770	2.0965	4986	2.0057	5206	1.9210	5430	1.8418	5658	1.7675	**30**
31	4773	2.0950	4989	2.0042	5209	1.9196	5433	1.8405	5662	1.7663	29
32	4777	2.0934	4993	2.0028	5213	1.9183	5437	1.8392	5665	1.7651	28
33	4780	2.0918	4997	2.0013	5217	1.9169	5441	1.8379	5669	1.7639	27
34	4784	2.0903	5000	1.9999	5220	1.9155	5445	1.8367	5673	1.7627	26
35	4788	2.0887	5004	1.9984	5224	1.9142	5448	1.8354	5677	1.7615	**25**
36	4791	2.0872	5008	1.9970	5228	1.9128	5452	1.8341	5681	1.7603	24
37	4795	2.0856	5011	1.9955	5232	1.9115	5456	1.8329	5685	1.7591	23
38	4798	2.0840	5015	1.9941	5235	1.9101	5460	1.8316	5688	1.7579	22
39	4802	2.0825	5019	1.9926	5239	1.9088	5464	1.8303	5692	1.7567	21
40	4806	2.0809	5022	1.9912	5243	1.9074	5467	1.8291	5696	1.7556	**20**
41	4809	2.0794	5026	1.9897	5246	1.9061	5471	1.8278	5700	1.7544	19
42	4813	2.0778	5029	1.9883	5250	1.9047	5475	1.8265	5704	1.7532	18
43	4816	2.0763	5033	1.9868	5254	1.9034	5479	1.8253	5708	1.7520	17
44	4820	2.0748	5037	1.9854	5258	1.9020	5482	1.8240	5712	1.7508	16
45	4823	2.0732	5040	1.9840	5261	1.9007	5486	1.8228	5715	1.7496	**15**
46	4827	2.0717	5044	1.9825	5265	1.8993	5490	1.8215	5719	1.7485	14
47	4831	2.0701	5048	1.9811	5269	1.8980	5494	1.8202	5723	1.7473	13
48	4834	2.0686	5051	1.9797	5272	1.8967	5498	1.8190	5727	1.7461	12
49	4838	2.0671	5055	1.9782	5276	1.8953	5501	1.8177	5731	1.7449	11
50	4841	2.0655	5059	1.9768	5280	1.8940	5505	1.8165	5735	1.7437	**10**
51	4845	2.0640	5062	1.9754	5284	1.8927	5509	1.8152	5739	1.7426	9
52	4849	2.0625	5066	1.9740	5287	1.8913	5513	1.8140	5743	1.7414	8
53	4852	2.0609	5070	1.9725	5291	1.8900	5517	1.8127	5746	1.7402	7
54	4856	2.0594	5073	1.9711	5295	1.8887	5520	1.8115	5750	1.7391	6
55	4859	2.0579	5077	1.9697	5298	1.8873	5524	1.8103	5754	1.7379	**5**
56	4863	2.0564	5081	1.9683	5302	1.8860	5528	1.8090	5758	1.7367	4
57	4867	2.0549	5084	1.9669	5306	1.8847	5532	1.8078	5762	1.7355	3
58	4870	2.0533	5088	1.9654	5310	1.8834	5535	1.8065	5766	1.7344	2
59	4874	2.0518	5092	1.9640	5313	1.8820	5539	1.8053	5770	1.7332	1
60	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	5774	1.7321	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	64°		63°		62°		61°		60°		′

′	30°		31°		32°		33°		34°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	5774	1.7321	6009	1.6643	6249	1.6003	6494	1.5399	6745	1.4826	**60**
1	5777	1.7309	6013	1.6632	6253	1.5993	6498	1.5389	6749	1.4816	59
2	5781	1.7297	6017	1.6621	6257	1.5983	6502	1.5379	6754	1.4807	58
3	5785	1.7286	6020	1.6610	6261	1.5972	6506	1.5369	6758	1.4798	57
4	5789	1.7274	6024	1.6599	6265	1.5962	6511	1.5359	6762	1.4788	56
5	5793	1.7262	6028	1.6588	6269	1.5952	6515	1.5350	6766	1.4779	**55**
6	5797	1.7251	6032	1.6577	6273	1.5941	6519	1.5340	6771	1.4770	54
7	5801	1.7239	6036	1.6566	6277	1.5931	6523	1.5330	6775	1.4761	53
8	5805	1.7228	6040	1.6555	6281	1.5921	6527	1.5320	6779	1.4751	52
9	5808	1.7216	6044	1.6545	6285	1.5911	6531	1.5311	6783	1.4742	51
10	5812	1.7205	6048	1.6534	6289	1.5900	6536	1.5301	6787	1.4733	**50**
11	5816	1.7193	6052	1.6523	6293	1.5890	6540	1.5291	6792	1.4724	49
12	5820	1.7182	6056	1.6512	6297	1.5880	6544	1.5282	6796	1.4715	48
13	5824	1.7170	6060	1.6501	6301	1.5869	6548	1.5272	6800	1.4705	47
14	5828	1.7159	6064	1.6490	6305	1.5859	6552	1.5262	6805	1.4696	46
15	5832	1.7147	6068	1.6479	6310	1.5849	6556	1.5253	6809	1.4687	**45**
16	5836	1.7136	6072	1.6469	6314	1.5839	6560	1.5243	6813	1.4678	44
17	5840	1.7124	6076	1.6458	6318	1.5829	6565	1.5233	6817	1.4669	43
18	5844	1.7113	6080	1.6447	6322	1.5818	6569	1.5224	6822	1.4659	42
19	5847	1.7102	6084	1.6436	6326	1.5808	6573	1.5214	6826	1.4650	41
20	5851	1.7090	6088	1.6426	6330	1.5798	6577	1.5204	6830	1.4641	**40**
21	5855	1.7079	6092	1.6415	6334	1.5788	6581	1.5195	6834	1.4632	39
22	5859	1.7067	6096	1.6404	6338	1.5778	6585	1.5185	6839	1.4623	38
23	5863	1.7056	6100	1.6393	6342	1.5768	6590	1.5175	6843	1.4614	37
24	5867	1.7045	6104	1.6383	6346	1.5757	6594	1.5166	6847	1.4605	36
25	5871	1.7033	6108	1.6372	6350	1.5747	6598	1.5156	6851	1.4596	**35**
26	5875	1.7022	6112	1.6361	6354	1.5737	6602	1.5147	6856	1.4586	34
27	5879	1.7011	6116	1.6351	6358	1.5727	6606	1.5137	6860	1.4577	33
28	5883	1.6999	6120	1.6340	6363	1.5717	6610	1.5127	6864	1.4568	32
29	5887	1.6988	6124	1.6329	6367	1.5707	6615	1.5118	6869	1.4559	31
30	5890	1.6977	6128	1.6319	6371	1.5697	6619	1.5108	6873	1.4550	**30**
31	5894	1.6965	6132	1.6308	6375	1.5687	6623	1.5099	6877	1.4541	29
32	5898	1.6954	6136	1.6297	6379	1.5677	6627	1.5089	6881	1.4532	28
33	5902	1.6943	6140	1.6287	6383	1.5667	6631	1.5080	6886	1.4523	27
34	5906	1.6932	6144	1.6276	6387	1.5657	6636	1.5070	6890	1.4514	26
35	5910	1.6920	6148	1.6265	6391	1.5647	6640	1.5061	6894	1.4505	**25**
36	5914	1.6909	6152	1.6255	6395	1.5637	6644	1.5051	6899	1.4496	24
37	5918	1.6898	6156	1.6244	6399	1.5627	6648	1.5042	6903	1.4487	23
38	5922	1.6887	6160	1.6234	6403	1.5617	6652	1.5032	6907	1.4478	22
39	5926	1.6875	6164	1.6223	6408	1.5607	6657	1.5023	6911	1.4469	21
40	5930	1.6864	6168	1.6212	6412	1.5597	6661	1.5013	6916	1.4460	**20**
41	5934	1.6853	6172	1.6202	6416	1.5587	6665	1.5004	6920	1.4451	19
42	5938	1.6842	6176	1.6191	6420	1.5577	6669	1.4994	6924	1.4442	18
43	5942	1.6831	6180	1.6181	6424	1.5567	6673	1.4985	6929	1.4433	17
44	5945	1.6820	6184	1.6170	6428	1.5557	6678	1.4975	6933	1.4424	16
45	5949	1.6808	6188	1.6160	6432	1.5547	6682	1.4966	6937	1.4415	**15**
46	5953	1.6797	6192	1.6149	6436	1.5537	6686	1.4957	6942	1.4406	14
47	5957	1.6786	6196	1.6139	6440	1.5527	6690	1.4947	6946	1.4397	13
48	5961	1.6775	6200	1.6128	6445	1.5517	6694	1.4938	6950	1.4388	12
49	5965	1.6764	6204	1.6118	6449	1.5507	6699	1.4928	6954	1.4379	11
50	5969	1.6753	6208	1.6107	6453	1.5497	6703	1.4919	6959	1.4370	**10**
51	5973	1.6742	6212	1.6097	6457	1.5487	6707	1.4910	6963	1.4361	9
52	5977	1.6731	6216	1.6087	6461	1.5477	6711	1.4900	6967	1.4352	8
53	5981	1.6720	6220	1.6076	6465	1.5468	6716	1.4891	6972	1.4344	7
54	5985	1.6709	6224	1.6066	6469	1.5458	6720	1.4882	6976	1.4335	6
55	5989	1.6698	6228	1.6055	6473	1.5448	6724	1.4872	6980	1.4326	**5**
56	5993	1.6687	6233	1.6045	6478	1.5438	6728	1.4863	6985	1.4317	4
57	5997	1.6676	6237	1.6034	6482	1.5428	6732	1.4854	6989	1.4308	3
58	6001	1.6665	6241	1.6024	6486	1.5418	6737	1.4844	6993	1.4299	2
59	6005	1.6654	6245	1.6014	6490	1.5408	6741	1.4835	6998	1.4290	1
60	6009	1.6643	6249	1.6003	6494	1.5399	6745	1.4826	7002	1.4281	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	59°		58°		57°		56°		55°		′

′	35° tan	35° cot	36° tan	36° cot	37° tan	37° cot	38° tan	38° cot	39° tan	39° cot	′
0	7002	1.4281	7265	1.3764	7536	1.3270	7813	1.2799	8098	1.2349	60
1	7006	1.4273	7270	1.3755	7540	1.3262	7818	1.2792	8103	1.2342	59
2	7011	1.4264	7274	1.3747	7545	1.3254	7822	1.2784	8107	1.2334	58
3	7015	1.4255	7279	1.3739	7549	1.3246	7827	1.2776	8112	1.2327	57
4	7019	1.4246	7283	1.3730	7554	1.3238	7832	1.2769	8117	1.2320	56
5	7024	1.4237	7288	1.3722	7558	1.3230	7836	1.2761	8122	1.2312	55
6	7028	1.4229	7292	1.3713	7563	1.3222	7841	1.2753	8127	1.2305	54
7	7032	1.4220	7297	1.3705	7568	1.3214	7846	1.2746	8132	1.2298	53
8	7037	1.4211	7301	1.3697	7572	1.3206	7850	1.2738	8136	1.2290	52
9	7041	1.4202	7306	1.3688	7577	1.3198	7855	1.2731	8141	1.2283	51
10	7046	1.4193	7310	1.3680	7581	1.3190	7860	1.2723	8146	1.2276	50
11	7050	1.4185	7314	1.3672	7586	1.3182	7865	1.2715	8151	1.2268	49
12	7054	1.4176	7319	1.3663	7590	1.3175	7869	1.2708	8156	1.2261	48
13	7059	1.4167	7323	1.3655	7595	1.3167	7874	1.2700	8161	1.2254	47
14	7063	1.4158	7328	1.3647	7600	1.3159	7879	1.2693	8165	1.2247	46
15	7067	1.4150	7332	1.3638	7604	1.3151	7883	1.2685	8170	1.2239	45
16	7072	1.4141	7337	1.3630	7609	1.3143	7888	1.2677	8175	1.2232	44
17	7076	1.4132	7341	1.3622	7613	1.3135	7893	1.2670	8180	1.2225	43
18	7080	1.4124	7346	1.3613	7618	1.3127	7898	1.2662	8185	1.2218	42
19	7085	1.4115	7350	1.3605	7623	1.3119	7902	1.2655	8190	1.2210	41
20	7089	1.4106	7355	1.3597	7627	1.3111	7907	1.2647	8195	1.2203	40
21	7094	1.4097	7359	1.3588	7632	1.3103	7912	1.2640	8199	1.2196	39
22	7098	1.4089	7364	1.3580	7636	1.3095	7916	1.2632	8204	1.2189	38
23	7102	1.4080	7368	1.3572	7641	1.3087	7921	1.2624	8209	1.2181	37
24	7107	1.4071	7373	1.3564	7646	1.3079	7926	1.2617	8214	1.2174	36
25	7111	1.4063	7377	1.3555	7650	1.3072	7931	1.2609	8219	1.2167	35
26	7115	1.4054	7382	1.3547	7655	1.3064	7935	1.2602	8224	1.2160	34
27	7120	1.4045	7386	1.3539	7659	1.3056	7940	1.2594	8229	1.2153	33
28	7124	1.4037	7391	1.3531	7664	1.3048	7945	1.2587	8234	1.2145	32
29	7129	1.4028	7395	1.3522	7669	1.3040	7950	1.2579	8238	1.2138	31
30	7133	1.4019	7400	1.3514	7673	1.3032	7954	1.2572	8243	1.2131	30
31	7137	1.4011	7404	1.3506	7678	1.3024	7959	1.2564	8248	1.2124	29
32	7142	1.4002	7409	1.3498	7683	1.3017	7964	1.2557	8253	1.2117	28
33	7146	1.3994	7413	1.3490	7687	1.3009	7969	1.2549	8258	1.2109	27
34	7151	1.3985	7418	1.3481	7692	1.3001	7973	1.2542	8263	1.2102	26
35	7155	1.3976	7422	1.3473	7696	1.2993	7978	1.2534	8268	1.2095	25
36	7159	1.3968	7427	1.3465	7701	1.2985	7983	1.2527	8273	1.2088	24
37	7164	1.3959	7431	1.3457	7706	1.2977	7988	1.2519	8278	1.2081	23
38	7168	1.3951	7436	1.3449	7710	1.2970	7992	1.2512	8283	1.2074	22
39	7173	1.3942	7440	1.3440	7715	1.2962	7997	1.2504	8287	1.2066	21
40	7177	1.3934	7445	1.3432	7720	1.2954	8002	1.2497	8292	1.2059	20
41	7181	1.3925	7449	1.3424	7724	1.2946	8007	1.2489	8297	1.2052	19
42	7186	1.3916	7454	1.3416	7729	1.2938	8012	1.2482	8302	1.2045	18
43	7190	1.3908	7458	1.3408	7734	1.2931	8016	1.2475	8307	1.2038	17
44	7195	1.3899	7463	1.3400	7738	1.2923	8021	1.2467	8312	1.2031	16
45	7199	1.3891	7467	1.3392	7743	1.2915	8026	1.2460	8317	1.2024	15
46	7203	1.3882	7472	1.3384	7747	1.2907	8031	1.2452	8322	1.2017	14
47	7208	1.3874	7476	1.3375	7752	1.2900	8035	1.2445	8327	1.2009	13
48	7212	1.3865	7481	1.3367	7757	1.2892	8040	1.2437	8332	1.2002	12
49	7217	1.3857	7485	1.3359	7761	1.2884	8045	1.2430	8337	1.1995	11
50	7221	1.3848	7490	1.3351	7766	1.2876	8050	1.2423	8342	1.1988	10
51	7226	1.3840	7495	1.3343	7771	1.2869	8055	1.2415	8346	1.1981	9
52	7230	1.3831	7499	1.3335	7775	1.2861	8059	1.2408	8351	1.1974	8
53	7234	1.3823	7504	1.3327	7780	1.2853	8064	1.2401	8356	1.1967	7
54	7239	1.3814	7508	1.3319	7785	1.2846	8069	1.2393	8361	1.1960	6
55	7243	1.3806	7513	1.3311	7789	1.2838	8074	1.2386	8366	1.1953	5
56	7248	1.3798	7517	1.3303	7794	1.2830	8079	1.2378	8371	1.1946	4
57	7252	1.3789	7522	1.3295	7799	1.2822	8083	1.2371	8376	1.1939	3
58	7257	1.3781	7526	1.3287	7803	1.2815	8088	1.2364	8381	1.1932	2
59	7261	1.3772	7531	1.3278	7808	1.2807	8093	1.2356	8386	1.1925	1
60	7265	1.3764	7536	1.3270	7813	1.2799	8098	1.2349	8391	1.1918	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	54°		53°		52°		51°		50°		′

′	40°		41°		42°		43°		44°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	8391	1.1918	8693	1.1504	9004	1.1106	9325	1.0724	9657	1.0355	60
1	8396	1.1910	8698	1.1497	9009	1.1100	9331	1.0717	9663	1.0349	59
2	8401	1.1903	8703	1.1490	9015	1.1093	9336	1.0711	9668	1.0343	58
3	8406	1.1896	8708	1.1483	9020	1.1087	9341	1.0705	9674	1.0337	57
4	8411	1.1889	8713	1.1477	9025	1.1080	9347	1.0699	9679	1.0331	56
5	8416	1.1882	8718	1.1470	9030	1.1074	9352	1.0692	9685	1.0325	55
6	8421	1.1875	8724	1.1463	9036	1.1067	9358	1.0686	9691	1.0319	54
7	8426	1.1868	8729	1.1456	9041	1.1061	9363	1.0680	9696	1.0313	53
8	8431	1.1861	8734	1.1450	9046	1.1054	9369	1.0674	9702	1.0307	52
9	8436	1.1854	8739	1.1443	9052	1.1048	9374	1.0668	9708	1.0301	51
10	8441	1.1847	8744	1.1436	9057	1.1041	9380	1.0661	9713	1.0295	50
11	8446	1.1840	8749	1.1430	9062	1.1035	9385	1.0655	9719	1.0289	49
12	8451	1.1833	8754	1.1423	9067	1.1028	9391	1.0649	9725	1.0283	48
13	8456	1.1826	8759	1.1416	9073	1.1022	9396	1.0643	9730	1.0277	47
14	8461	1.1819	8765	1.1410	9078	1.1016	9402	1.0637	9736	1.0271	46
15	8466	1.1812	8770	1.1403	9083	1.1009	9407	1.0630	9742	1.0265	45
16	8471	1.1806	8775	1.1396	9089	1.1003	9413	1.0624	9747	1.0259	44
17	8476	1.1799	8780	1.1389	9094	1.0996	9418	1.0618	9753	1.0253	43
18	8481	1.1792	8785	1.1383	9099	1.0990	9424	1.0612	9759	1.0247	42
19	8486	1.1785	8790	1.1376	9105	1.0983	9429	1.0606	9764	1.0241	41
20	8491	1.1778	8796	1.1369	9110	1.0977	9435	1.0599	9770	1.0235	40
21	8496	1.1771	8801	1.1363	9115	1.0971	9440	1.0593	9776	1.0230	39
22	8501	1.1764	8806	1.1356	9121	1.0964	9446	1.0587	9781	1.0224	38
23	8506	1.1757	8811	1.1349	9126	1.0958	9451	1.0581	9787	1.0218	37
24	8511	1.1750	8816	1.1343	9131	1.0951	9457	1.0575	9793	1.0212	36
25	8516	1.1743	8821	1.1336	9137	1.0945	9462	1.0569	9798	1.0206	35
26	8521	1.1736	8827	1.1329	9142	1.0939	9468	1.0562	9804	1.0200	34
27	8526	1.1729	8832	1.1323	9147	1.0932	9473	1.0556	9810	1.0194	33
28	8531	1.1722	8837	1.1316	9153	1.0926	9479	1.0550	9816	1.0188	32
29	8536	1.1715	8842	1.1310	9158	1.0919	9484	1.0544	9821	1.0182	31
30	8541	1.1708	8847	1.1303	9163	1.0913	9490	1.0538	9827	1.0176	30
31	8546	1.1702	8852	1.1296	9169	1.0907	9495	1.0532	9833	1.0170	29
32	8551	1.1695	8858	1.1290	9174	1.0900	9501	1.0526	9838	1.0164	28
33	8556	1.1688	8863	1.1283	9179	1.0894	9506	1.0519	9844	1.0158	27
34	8561	1.1681	8868	1.1276	9185	1.0888	9512	1.0513	9850	1.0152	26
35	8566	1.1674	8873	1.1270	9190	1.0881	9517	1.0507	9856	1.0147	25
36	8571	1.1667	8878	1.1263	9195	1.0875	9523	1.0501	9861	1.0141	24
37	8576	1.1660	8884	1.1257	9201	1.0869	9528	1.0495	9867	1.0135	23
38	8581	1.1653	8889	1.1250	9206	1.0862	9534	1.0489	9873	1.0129	22
39	8586	1.1647	8894	1.1243	9212	1.0856	9540	1.0483	9879	1.0123	21
40	8591	1.1640	8899	1.1237	9217	1.0850	9545	1.0477	9884	1.0117	20
41	8596	1.1633	8904	1.1230	9222	1.0843	9551	1.0470	9890	1.0111	19
42	8601	1.1626	8910	1.1224	9228	1.0837	9556	1.0464	9896	1.0105	18
43	8606	1.1619	8915	1.1217	9233	1.0831	9562	1.0458	9902	1.0099	17
44	8611	1.1612	8920	1.1211	9239	1.0824	9567	1.0452	9907	1.0094	16
45	8617	1.1606	8925	1.1204	9244	1.0818	9573	1.0446	9913	1.0088	15
46	8622	1.1599	8931	1.1197	9249	1.0812	9578	1.0440	9919	1.0082	14
47	8627	1.1592	8936	1.1191	9255	1.0805	9584	1.0434	9925	1.0076	13
48	8632	1.1585	8941	1.1184	9260	1.0799	9590	1.0428	9930	1.0070	12
49	8637	1.1578	8946	1.1178	9266	1.0793	9595	1.0422	9936	1.0064	11
50	8642	1.1571	8952	1.1171	9271	1.0786	9601	1.0416	9942	1.0058	10
51	8647	1.1565	8957	1.1165	9276	1.0780	9606	1.0410	9948	1.0052	9
52	8652	1.1558	8962	1.1158	9282	1.0774	9612	1.0404	9954	1.0047	8
53	8657	1.1551	8967	1.1152	9287	1.0768	9618	1.0398	9959	1.0041	7
54	8662	1.1544	8972	1.1145	9293	1.0761	9623	1.0392	9965	1.0035	6
55	8667	1.1538	8978	1.1139	9298	1.0755	9629	1.0385	9971	1.0029	5
56	8672	1.1531	8983	1.1132	9303	1.0749	9634	1.0379	9977	1.0023	4
57	8678	1.1524	8988	1.1126	9309	1.0742	9640	1.0373	9983	1.0017	3
58	8683	1.1517	8994	1.1119	9314	1.0736	9646	1.0367	9988	1.0012	2
59	8688	1.1510	8999	1.1113	9320	1.0730	9651	1.0361	9994	1.0006	1
60	8693	1.1504	9004	1.1106	9325	1.0724	9657	1.0355	1.000	1.0000	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	49°		48°		47°		46°		45°		′

ANSWERS

Ex. 1:

1. 7 1/3 ft.	*2.* 43′2″	*3.* 23,760 ft. per min.
4. 184.5 sq. ft.	*5.* 60 mi. per hr.	*6.* 10 gal.
7. $3.36	*8.* 20 ft./sec.	*9.* 69.4 sq. ft.
10. .134 cu. ft.; 4.2 cu. ft.	*11.* 8.32 lb.	*12.* 12,480 lb.
13. 40 cu. ft.	*14.* 84,000 lb.	*15.* 600 cu. ft./min.

Ex. 2:

1. 8.75 oz.	*2.* 1.8 in.	*3.* 37.5 cm.	*4.* 5.4 kg.	*5.* 20″
6. 165 lb.	*7.* 162.5 mm.	*8.* 226.8 gm.	*9.* 77.4 kg.	*10.* 5′11.2″
11. 1900 cc.	*12.* 10.6 qt.	*13.* 1.59 pt.	*14.* 7.6 l.	*15.* 1300 sq. cm.

Ex. 3:

1. 24/64″; 36/16″; 52/32″; 652/64″	*2.* 4 1/2 eighths; 1/16″ more; 3/16″ less
3. 40/16; 26/16; 51/16; 12/16	*4.* 23
5. 58/16; 116/32	*6.* No; yes; 3/4, 7/8, 2 3/64, 2 1/16

Ex. 4:

1. 1 5/12	*2.* 1 3/8	*3.* 1 3/4	*4.* 15/16	*5.* 8 3/8
6. 6 15/16	*7.* 8 5/16	*8.* 8 17/32	*9.* 2 3/8	*10.* 2 3/16
11. 7/64	*12.* 1 15/32	*13.* 5 29/32″	*14.* 29 5/8″	*15.* 80 1/4″
16. 1 5/64″	*17.* 1 1/8″; 2 1/16″; 11/16″	*18.* 5 5/8″; 5/8″	*19.* 17 1/16″	*20.* 10 5/16″

Ex. 5:

1. 3 7/16	*2.* 87	*3.* 78	*4.* 5 1/2	*5.* 8	*6.* 5/8
7. 160.3 gal.	*8.* 5 11/16″	*9.* 2 3/4″	*10.* 19	*11.* 282 3/4 lb.	*12.* 55

Ex. 6:

1. 0.276 *2.* 0.0154 *3.* 0.0007 *4.* 0.00425 *5.* 0.100

6. Three hundred seventy-nine and two-tenth thousandths.
7. Six-tenth thousandths.
8. Two hundred and two-tenth thousandths.
9. Seventy and one-half thousandths.
10. Two hundred eighty-one and six-tenth thousandths.
11. Ninety-six thousandths.
12. Four hundred forty-four and four-tenth thousandths.
13. Fifteen and eight-tenth thousandths.
14. One-quarter thousandth.

Ex. 7:

1. 2.5175″	*2.* 2.3758″	*3.* 1.192″	*4.* .6175″
5. .694″	*6.* 4.3917″	*7.* 1.372″	*8.* 4.214″; 1.81″
9. 1.503″; 3.678″	*10.* .0007″	*11.* .239″	*12.* 1.444″; 2.786″; .7883″

Ex. 8:

1. 5.95″ *2.* 445.09 lb. *3.* .682″ *4.* 216.5 lb. *5.* .108″
6. 7.57 lb. *7.* .648″ *8.* 139.7 cal. *9.* $336.86 *10.* .10752″

Ex. 9:

1. 73.5 cu. ft. *2.* 30.5 cu. ft. *3.* 16.61 cm. *4.* 36 *5.* 5.438″ *6.* 96

Ex. 10:

1. .125 *2.* .09375 *3.* .4375 *4.* .234375 *5.* .28125
6. .9375 *7.* .84375 *8.* .6875 *9.* 3.142857 *10.* .555556
11. .9230769 *12.* .181818 *13.* .171875 *14.* .765625 *15.* .208333
16. .19531 *17.* 7/16 *18.* 13/16 *19.* 1/16 *20.* 1/320
21. 13/32 *22.* 29/32 *23.* 5/64 *24.* 59/64 *25.* 17/32

Ex. 11:

1. (*a*) 7/32 (*b*) 27/32 (*c*) 3/32 *2.* (*a*) 31/64 (*b*) 11/64 (*c*) 51/64
3. 1.3866 to 1.3946; 6.246 to 6.254; .961 to .969; 1.038 to 1.046
4. .88125 to .93125; 1.13125 to 1.18125; 1.1625 to 1.2125; .5375 to .5875; 2.100 to 2.150

Ex. 12:

1. .908″; .035″ *2.* $x=2\frac{5}{8}''$, $y=2\frac{1}{4}''$; $x=2.640''$, $y=2.2575''$
3. 13/16″, 2 11/16″, 3 5/32″, 1 3/16″ *4.* .762″; 2.637″; 3.258″; 1.141″
5. At least .003″, but not more than .007″

Ex. 13:

1. (*a*) .465 (*b*) .904 (*c*) .595 (*d*) .132 (*e*) .022 (*f*) .991
2. (*a*) .3473 (*b*) .8839 (*c*) .4662 (*d*) .2114 (*e*) .5776 (*f*) .6241

Ex. 14:

1. .1004″, .144″, .150″ *2.* .1002″, .350″, .103″
3. .1005″, .200″, .126″ *4.* .550″, .116″
5. .1002″, .150″; .108″, .040″ *6.* .1008″, .200″, .133″
7. 3.000″, .1001″, .700″, .106″ *8.* 2.000″, .1002″, .500″, .107″
9. 1.000″, .1009″, .650″, .103″
10. 4.000″, 3.000″, 1.000″, .900″, .500″, .140″, .1002″
11. 4.000″, 3.000″, .144″, .050″, .1004″ *12.* 3.000″, .1008″, .800″, .109″

Ex. 15:

1. 57.22 lb. *2.* 94.6 H.P. *3.* 2769 r.p.m.
4. 90 H.P. *5.* 7½ lb. *6.* $12.04; $8.74
7. 21 lb.; 40½ lb.; 88½ lb. *8.* 27.5 lb. *9.* 6 pieces
10. 4.728″ and 4.872″ *11.* 217½ lb.; 797½ lb.; 435 lb.

Ex. 16:

1. 16⅔% *2.* 15% *3.* 12% *4.* 25.7%
5. 4.5% *6.* 12½% *7.* 40% *8.* 1.6%
9. .0332% *10.* 25% *11.* 1.8% *12.* 8.6%

Ex. 17:

1. 156; 195 *2.* 22.5″ *3.* 1000 ft. *4.* 88 lb. *5.* $160,000 *6.* $48.50

Ex. 18:

1. $10.80 *2.* $1.60 *3.* 20% *4.* $70 *5.* 12½%
6. $22,400 *7.* 3.6 oz. *8.* 6⅔% *9.* 390 cu. ft. *10.* 3⅛%
11. 368 *12.* 4.2% *13.* $560 *14.* $115.63 *15.* $120

Ex. 19:

1. 2:3 *2.* 1:6 *3.* 2:3
4. 8:19 *5.* 13:17 *6.* 17½″
7. 15:23 *8.* 7 in. *9.* .04 cm.
10. 7 in.

Ex. 20:

1. 5:2; 2:5 *2.* 943:1000 *3.* 17½″, 24½″ *4.* 30°, 60°, 90°
5. 19 liters *6.* 17.32″; 3″ *7.* 1:4; 55 *8.* 200.26 cm.
9. 18 lb. *10.* 300, 700, 1000 *11.* .88 *12.* 460 lb.
13. 1:4 *14.* 2:5; 5:3; 30% *15.* 5:8; 2:5; 33⅓%

Ex. 21:

1. 1″×2⅛″ *2.* 2½″×1¾″
3. 8″; 20″; 15′; 6½″ *4.* 2½′; 1″; ¼″; 1.2″; 40′; 72½′
5. (*a*) 1¾″; 65′ (*b*) 3″; 60″ (*c*) 1½″; 19′ (*d*) 12½″; 11½ mi.
(*e*) 1⅛″; 180 ft.
6. 14″×24½″ *7.* 3¼″×4½″ *8.* 8.17″ *9.* 14′×23′ *10.* 27′×37½′
11. 37½ ft. *12.* 2½″
13. 1″=1′; 1″=10′; 3″; 12′; 10″; 1″=2′ *14.* 3½′×7′
15. L.R., 12′×20′; Dinette, 9½′×10′; B.R., 11½′×17′; Alcove, 8½′×12½′

Ex. 22:

1. 20 *2.* 40 *3.* 21 *4.* 20 *5.* 12 *6.* 27
7. 9 *8.* 42 *9.* 6 *10.* 24 *11.* 6 *12.* 28

Ex. 23:

1. 1.8″ *2.* $56 *3.* 525 pieces *4.* 2380 mi.
5. 525 ohms *6.* 780 lb. *7.* 42 ft.; 250 lb. *8.* 326⅔; 642.9
9. 288 cu. ft.; 125 lb. *10.* .56 mi. nearer

Ex. 24:

1. $C_2=12$; $R_2=60$ *2.* 576 cu. in.; 72 lb. per sq. in. *3.* $w_2=3$ lb.; $d_2=8''$
4. .0024 *5.* 24 r.p.m. *6.* 24″
7. 700 r.p.m. *8.* 13 teeth *9.* 2500 r.p.m.
10. 27″

Ex. 25:

1. Twice one number increased by three times another number.
2. Five times one number diminished by half another number.
3. One number decreased by a second and increased by a third, and the result divided by three.
4. Two-thirds of a number increased by eight.
5. One number decreased by the reciprocal of another number.
6. Five times the product of two numbers.
7. The difference between two numbers divided by twice a third number.
8. Nine-fifths of a number increased by thirty-two.
9. Twice the product of the length by the width added to twice the product of the length and height, added to twice the product of the width and height.
10. The difference of two numbers divided by the difference of two other numbers.

Ex. 26:

1. 11.4
2. 10.33
3. 90
4. 3.586
5. 179.7
6. .0689
7. 211.1
8. 2, 3, 4, 5, 7½
9. 0, 8, 16, 24, 32
10. 3, 6, 7½, 9, 18
11. 0, ¼, ½, ¾, 1¼
12. 2⅔, 5⅓, 8, 10⅔, 16
13. ⅔, 2, 3⅓, 4⅔, 6⅔
14. 0, .32, .64, 3.19
15. 0, 4½, 9, 18, 45

Ex. 27:

1. +6; +38; 0; —9; —22; —12; +12
2. +23; +13; —6; 0; —2; —36; 0
3. —63; +60; —60; +2; +6.2; $-72k$; $+30m$; $-12a$; $-48r$
4. —9; —9; +3; +12; —2; +1; —2; $-8/a$; $-8x$

Ex. 28:

1. $21p$; $26r$; $9m$; $8.6h$; $-22\frac{1}{2}x$
2. $4k$; $-6a$; $26xy$; $-36lw$; $+2\frac{1}{2}ab$
3. $34x$
4. $26k$
5. $4h+4l+4w$
6. $12e$
7. $8\frac{1}{4}k$
8. $9d$

Ex. 29:

1. $32m$; $30k$; $15pq$; $12abh$; $12a^3b$; $66h$; $\frac{1}{3}st$; $5r^2h$; $\frac{4}{3}R^3$; $7.2a^2t^3$; $12a^3b^3$; $5x^7$; m^5; $6x^2y^2$; $\frac{1}{6}a^5$
2. $7m$; 8; 24; $3/x$; $7\frac{1}{2}k$; $7y$; $12a^2/b$; 5; $3r/s^3$; $-8d^2/c$; $2Rh$; $-2a^2b^2$

Ex. 30:

1. $2l+2w$
2. $ka+kb-kc$
3. $a+nd-d$
4. $IR+Ir$
5. $P+PRT$
6. $S=\frac{na}{2}+\frac{nl}{2}$
7. $\frac{hB}{2}+\frac{hb}{2}$
8. $\pi R^2-\pi r^2$
9. Lt_0-Lt_1
10. $2\pi rl+2\pi r^2$
11. $na+\frac{n^2d}{2}-\frac{nd}{2}$
12. $V=\frac{hb_1}{6}+\frac{hb_2}{6}+\frac{2}{3}hM$
13. $p\ (1+i)$
14. $h\ (p-q)$
15. $I^2\ (R+r)$
16. $2a\ (b-c)$
17. $P\ (1+rt)$
18. $\frac{1}{2}h\ (B_1+B_2)$
19. $\pi D\ (l+1)$
20. $2\pi R\ (l+R)$
21. 100; 20
22. 21.58
23. .5625
24. 8448

Ex. 31:

1. 64
2. 256
3. 216
4. 1728
5. 1
6. 27
7. 400
8. 343
9. ¼
10. 4/9
11. 1/64
12. 1/100
13. 27/64
14. .0001
15. .000001
16. .000001

Ex. 32:

1. 3,400,000
2. 6,200,000,000
3. 91,000,000,000
4. 430,000,000,000
5. 5,200,000
6. 1,380,000,000,000
7. 862,000,000
8. 24,630,000,000,000
9. .000035
10. .00000023
11. .0000000049
12. .000000000326
13. 37×10^{6}
14. 5.8×10^{6}
15. 12.4×10^{6}
16. 4.9×10^{9}
17. 4×10^{-5}
18. 2.8×10^{-7}
19. 3.92×10^{-4}
20. 7.6×10^{-10}
21. 5,882,500,000,000
22. 58.93×10^{-5}

Ex. 33:

1. $\sqrt[3]{a}$; $\sqrt[4]{x}$; $\sqrt{5a}$; $\sqrt[3]{p^2}$; $\sqrt{A^3}$; $\sqrt{mn}$
2. $p^{1/2}$; $k^{1/4}$; $a^{2/3}$; $(10x)^{1/6}$; $\left(\frac{2s}{g}\right)^{1/2}$; $\left(\frac{A}{\pi}\right)^{1/2}$
3. 12; 1½; 8; 4; $6a$; 12; 216; 4,000,000
4. ⅕; 8; 9; .2; 16; 2; .3; 8; ½; .01; $\frac{1}{25x^2}$; $\frac{1}{4a^2}$

Ex. 34:

1. 364 in.
2. 37½ in.
3. 456
4. 64.4°
5. $510
6. 2.45
7. 200,000
8. 15.84
9. 59,040
10. 26.4 cal.
11. 153.6 H.P.
12. 128,060 sq. ft.

Ex. 35:

1. 13
2. 32
3. .35
4. 20
5. 12
6. 15
7. 14
8. 320
9. 12
10. 36
11. 16
12. 160

Ex. 36:

1. $l=\frac{A}{w}$
2. $M=DV$
3. $T=\frac{D}{R}$
4. $R=\frac{E}{I}$
5. $R=\frac{C}{2\pi}$
6. $s=\frac{d}{1.4}$
7. $P=\frac{kT}{V}$
8. $T=\frac{I}{PR}$
9. $h=\frac{2A}{b}$
10. $r=\frac{S}{2\pi h}$
11. $w_2=\frac{l_1w_1}{l_2}$
12. $t^2=\frac{2S}{g}$

Ex. 37:

1. 22
2. 14
3. 38
4. 95
5. 14
6. 36
7. 3½
8. 10.7
9. 2.1
10. 9.25
11. 22.5
12. 38.3

Ex. 38:

1. 2 2. 4 3. $\frac{n-p}{k}$ 4. $\frac{C-q}{4}$

5. $\frac{P-2l}{2}$ 6. $\frac{V-E}{r}$ 7. $\frac{12D-TL}{12}$ 8. $\frac{V-v_0}{g}$

9. $\frac{2A-bh}{h}$ 10. $\frac{A-P}{PT}$ 11. $\frac{l-a}{n-1}$ 12. $N-2PC$

13. $\frac{DL}{2\pi}$ 14. $NW+W; \frac{s}{N+1}$ 15. $\frac{FRg}{v^2}; \frac{Wv^2}{gF}$ 16. $\frac{Q(t_1-t_0)+wh}{w}$

Ex. 39:

1. 4.2 2. 5.5 3. 3.5 4. 6.4 5. 7.7
6. 4.6 7. 7.4 8. 8.6 9. 11.4 10. 12.2
11. 13.1 12. 14.1 13. 9.3 14. 7.4 15. 2.5
16. 19.5 17. 23.8 18. 54.2 19. 8.3 20. 20.7

Ex. 40:

1. 7.4 2. 7.6 3. 13.3 4. 32.0
5. 6.4 6. 11.22 7. 166.7 8. 35.4 mi.
9. 40 sq. ft. 10. 9.2 in.

Ex. 41:

1. $v=\sqrt{\frac{2E}{m}}$ 2. $t=\sqrt{\frac{2v}{g}}$ 3. $h=\frac{2m^2}{3}$

4. $h=\frac{s}{2}\sqrt{3}$ 5. $s=\sqrt{\frac{4A}{\sqrt{3}}}$ 6. $R=\sqrt{\frac{3V}{\pi h}}$

7. $C=\frac{1}{4\pi^2 f^2 L}$ 8. $C=\sqrt{\frac{6I}{\pi ab}}$ 9. $v=10\sqrt{\frac{11gP}{W}}$

10. $R=\frac{1}{C}\sqrt{E^2-(2\pi nLC)^2}$

11. $h=\frac{.36P^2}{A^2}$ 12. 38.0 sq. ft.

Ex. 42:

1. 2.7; —1.4 2. 12.4; —2.4 3. 3.3; —4.3
4. 1.5; —1.3 5. 3.8; —.3 6. 3.0; —5.0

Ex. 43:

1. Doubled; halved; divided by 10; multiplied by 1½; divided by 8
2. Multiplied by 9; divided by 6; constant; multiplied by 4
3. *V* decreases; *P* decreases; *P* is doubled; *V* is divided by 3; *P* is divided by 5
4. Multiplied by 4; divided by 9; multiplied by 25; divided by 100

Ex. 48:

1. 2.9694 *2.* 2.6112 *3.* 9.7024—10 *4.* 1.8554
5. 6.7210—10 *6.* 5.8082 *7.* .9051 *8.* 7.5328—10
9. 9.8408—10 *10.* 6.6532—10 *11.* 3.6817 *12.* 2.3746

Ex. 49:

1. 24.02 *2.* 4.397 *3.* 265.3 *4.* .4116 *5.* 25.06 *6.* .1996
7. 5.788 *8.* .02619 *9.* 276.4 *10.* 52,663 *11.* .008134 *12.* 1112

Ex. 50:

1. 1.333 *2.* .00986 *3.* 1051 *4.* 4185 *5.* .2374 *6.* .004759

Ex. 51:

1. 9.110 *2.* 3.894 *3.* .05604 *4.* 2.116 *5.* 15.78
6. 2.222 *7.* 1.486 *8.* 65,000 *9.* .01985 *10.* 19,620
11. 465.8 *12.* 10.72 *13.* 2.466 *14.* 4.444 *15.* 5.46 in.

Ex. 52:

1. 1.796 *2.* 13.75 *3.* 1.181 *4.* 4.166 *5.* 2.56
6. 1.14 *7.* 1.79 *8.* 47.9 *9.* 18.3 yr. *10.* $678.70; 4.3 yr.

Ex. 53:

1. 950 *2.* 565 *3.* 57.4 *4.* .231 *5.* 123.5 *6.* 116.1 *7.* 710
8. 4.03 *9.* 38.3 *10.* 4.16 *11.* 79.2 *12.* 1.44 *13.* 5.21 *14.* 5.12

Ex. 54:

1. 6.63 *2.* 4.98 *3.* 8.4 *4.* 11.7 *5.* 2.87 *6.* .92
7. .0179 *8.* .253 *9.* 58.7 *10.* .07 *11.* 1.75 *12.* .087

Ex. 55:

1. 51°; 67½°; 45°40′; 22°18′48″
2. 148°; 135°; 90°; 58°; 122°31′; 19°15′12″
3. 85°20′ *4.* 100° *5.* 85° *6.* 56°; 34° *7.* 34° *8.* 50°

Ex. 56:

1. 142°; 68°; 25°
2. $x=14°$; $y=166°$; $z=76°$
3. 120°; 58°; 360°
4. 115°; 68°; 130°; 68°; 132°
5. 64°45′
6. 133°
7. AM, MC, AB; BC, BM, BM
8. OR, point O, OS; OM, OB, OP

Ex. 57:

1. 75° 24′ *2.* 16° *3.* 45° 27′ *4.* 28° 35′ 50″ *5.* 64° 38′ *6.* 109° 12′

Ex. 58:

1. 12.04 in. *2.* 20.6 ft. *3.* 11.8 in. *4.* 6.4 in. *5.* 22.4 in.
6. 8.9 in. *7.* 7.2 in.; 10.8 in. *8.* 22′ 9″ *9.* 5.66 in. *10.* 9.798 in.

Ex. 59:

1. 200° *2.* 215° *3.* 158° *5.* $\widehat{AB}=40°$; $\widehat{DC}=40°$; $\widehat{BC}=220°$
6. $\angle P=95°$; $\angle Q=65°$; $\angle R=85°$; $\angle S=115°$

Ex. 60:

1. 15.5 in. *2.* 17° 10′; $\alpha=\frac{\beta}{2}$ *3.* 9.899 in.
4. 4.58 in.; $x=\frac{1}{2}d\sqrt{2}$ *5.* 220 ft. per sec. *6.* 160 r.p.m.
7. 9.05 in. *8.* $\phi=57°\ 30'$; $\phi=2\alpha$ *9.* 14.14″; 44.44″
10. 288 times *11.* 1.225″; $y=s\sqrt{2}$ *12.* Twice as long
13. 5.66″ *14.* 54° *15.* 6″
16. 1½ revolutions *17.* $k=\frac{D}{2}\sqrt{3}$; 2.70″ *18.* 50.3 ft.
19. 1.06″ *20.* 2263 ft. per min. *21.* 3 in.
22. 13.13 in. *23.* 17.34 in. *24.* 23.6 in.
25. 110 ft.

Ex. 64:

1. 42.4 sq. in. *2.* 22.2 sq. in. *3.* 61.2 sq. in.
4. 288 sq. in. *5.* 2 sq. in. *6.* 396 sq. ft.
7. 36 sq. in. *8.* 26.5 sq. in. *9.* 896 sq. ft.
10. 176 sq. in. *11.* 2400 sq. rd.; 15A. *12.* 320 sq. ft.
13. 56 sq. in. *14.* 17.8 sq. in. *15.* 2.59 sq. in.

Ex. 65:

1. 3.14 sq. in. *2.* 10.63 in. *3.* 366.7 sq. in.
4. 245.6 sq. in. *5.* 2:3; 4:9 *6.* 17.3%
7. 6.48 sq. in. *8.* 22.2 sq. in. *9.* 36.5%
10. 7.2 sq. in. *11.* 3.4 sq. in. *12.* 10.1 sq. in.
13. 484 sq. in. *14.* 50.3 sq. in. *15.* 21,384 lb.
16. 3 in. *17.* 40% *18.* 10 in.
19. 12.57 sq. in.; 11.78 sq. in. *20.* 209.3 sq. in.

Ex. 66:

1. 72 lb. *2.* 136 sq. ft. *3.* 46.1 cu. ft.
4. 6.29 cu. in.; 1.95 lb. *5.* 24.75 cu. in. *6.* $27.20
7. 38.29 lb. *8.* 450 sq. ft. *9.* 12½ lb.
10. 37.4 gal. *11.* 125 *12.* 11.7 cu. in.
13. 28.9 cu. in. *14.* 156.8 cu. in. *15.* 2282.8 ft.
16. 6.9 cu. yd. *17.* 12 in. *18.* 5,715 lb.
19. 6.48 in. *20.* 6,594 sq. ft. *21.* 29.2 lb.
22. 33,880 cu. yd.

Ex. 67:

1. 4.8 gal. *2.* 14.5 cu. yd. *3.* 3,014 cu. cm. *4.* 33⅓%
5. 847.8 sq. in. *6.* 242.5 cu. in. *7.* 187.7 cu. ft. *8.* 44.9 cu. yd.

9. 990 sq. in. *10.* 7.07 cu. ft. *11.* 87.5 sq. cm. *12.* 268.7 cu. in.
13. 200.8 cu. in. *14.* 1:4 *15.* 122,033 gal. *16.* 3.6 cu. in.

Ex. 68:

1. 80 in. *2.* 108 sq. ft. *3.* BD=8.94; AB=12.00
4. AC=15; BC=20 *5.* 135 ft. *6.* (a) 3.2″, 12.8″; (b) 16″
7. (a) 7.2″; (b) 9.6″; 5.4″ *8.* 12″ and 15″, respectively

Ex. 69:

1. Half; one-fourth *2.* 16π, or 50.24″
3. Area, 4 times; volume, 8 times *4.* 3:4; 9:16
5. 28.28 in.
6. Circumference, 10% less; area, 19% less
7. 10 in. *8.* 9:16 *9.* 125% *10.* 5.19 to 1
11. L.A.=401.9 sq. in.; V=927.9 cu. in. *12.* 56.25% increase
13. T.A.=195.7 sq. in.; V=169.4 cu. in. *14.* 12 sq. in.
15. 72.8% *16.* 125%
17. Twice as large *18.* 400 times larger
19. 4 times; 4 times *20.* 125%

Ex. 70:

1. 26.3 in. *2.* 18° 26′ *3.* 12.16; 80° 32′; 18° 58′
4. 176.3 ft. *5.* 4.83 *6.* 36° 52′

Ex. 71:

1. 24.8 in. *2.* 40.0″; 24° 43′; 130° 34′ *3.* 21.47″; 67° 58′
4. 3.39-inch square *5.* 71° 47′; 36° 26′; 15.2 in. *6.* 7.46″; 7.73″
7. 10.77″; 81° 18′ *8.* 3.696″ *9.* 4.33″
10. 12.8″; 15.8″ *11.* 10.77′; 27.58′ *12.* 59° 30′; 1.89″
13. .343″ *14.* 1.805″ *15.* 92° 48′

Ex. 72:

1. 167.7 ft. per min. *2.* 212.13 mi. per hr. N.; same for E.
3. 79.5 lb.; 127.2 lb. *4.* 459.6 lb.; 385.7 lb.
5. 456.75 lb.; 203.35 lb. *6.* 45.3 ft. per sec.; 21.3 ft. per sec.
7. 1000 lb.; 36° 52′ with the horizontal *8.* 176.78 lb.

Ex. 73:

1. 5.358″ *2.* 10.39″ *3.* 4.50″ *4.* 20.5″

Ex. 74:

1. 5° 12′ *2.* 26° 59′ *3.* 33°; .5924″
4. 33° 40′ *5.* 2.5584″ *6.* 25° 22′; 12° 50′

Ex. 75:

1. 6.96″ *2.* 3.33″ *3.* 5.5″ *4.* 1.46″
5. 4.402″ *6.* 2.55″; 3.56″ *7.* .048″ *8.* 39° 58′
9. 1.06″ *10.* .96″ *11.* 73° 44′ *12.* .65″

Ex. 76:

1. $a=13.6''$; $b=16.4''$; $C=82° \ 50'$
2. $a=31.4''$; $c=46.1''$; $B=28° \ 50'$
3. $b=25.7''$; $c=28.5''$; $C=79° \ 3'$
4. $b=11.79''$; $A=94° \ 5'$; $C=38° \ 35'$
5. $c=29.72''$; $A=28° \ 25'$; $B=16° \ 35'$

Ex. 77:

1. $c=30.4''$; $C=82° \ 32'$; $B=31° \ 28'$
2. $A=31° \ 35'$; $B=38° \ 57'$; $C=109° \ 28'$
3. $a=4.3''$; $A=46° \ 15'$; $C=23° \ 45'$
4. $A=55° \ 17'$; $B=39° \ 37'$; $C=85° \ 6'$
5. $b=21.5''$; $B=96° \ 17'$; $C=56° \ 13'$

Ex. 78:

1. 115.97 sq. in.
2. 430.98 sq. in.
3. 1175.6 sq. cm.
4. 4.26 in.
5. 403.1 sq. in.
6. $x=17° \ 21'$; $y=107° \ 21'$
7. 5.75 in.
8. 40.9 ft.
9. 1.53 in.
10. 82° 6′; 97° 54′

INDEX